HOMOLOGY AND COHOMOLOGY OF DIAGRAMS

OF TOPOLOGICAL SPACES

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HOMOLOGY AND COHOMOLOGY OF DIAGRAMS OF TOPOLOGICAL SPACES

A

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ABSTRACT

Homology and cohomology of objects other than ordinary topological spaces have been investigated by several authors. Let X be a G–space and $\mathcal{F}$ be a family of all closed subgroups of G. The equivariant cellular structures are obtained by attaching n–cells of the form $G/H \times B^n$, where B^n is the unit n–ball and $H \in \mathcal{F}$. A construction of the equivariant singular homology and cohomology for X and $\mathcal{F}$ was given by Sören Illman. In this thesis $\mathcal{F}$ is replaced by a small topological category and the G–space X is replaced by a functor taking values in *k*–spaces and called a diagram of topological spaces. The cellular structures for diagrams are obtained by attaching cells $D_j \times B^n$ where D_j is a representable functor.

The purpose of this thesis is to investigate homology and cohomology for diagrams. We review the foundation of homotopy theory and cellular theory for diagrams. We propose axioms for homology and cohomology modeled on Eilenberg–Steenrod system of axioms and prove that they, as in the classical case, determine uniquely homology and cohomology for finite cellular diagrams. We give the generalization of Illman's equivariant singular homology and cohomology to diagrams of topological spaces and we prove that this generalization satisfies all introduced axioms. Also we prove the comparison theorem between the sheaf cohomology for diagrams developed by Robert J. Piacenza and the singular cohomology for diagrams developed in this thesis.

TABLE OF CONTENTS

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CHAPTER 0

INTRODUCTION

0.1 Diagrams of Topological Spaces

Homology theory and homotopy theory are seen as the oldest and most extensively developed portions of algebraic topology. The typical task in homology theory is to assign a graded group $H_*(X; \mathbb{K})$ to a pair $(X, \mathbb{K})$, where X is a topological space and $\mathbb{K}$ is an abelian group. The space X can have a simple or very rich structure, it can be a polyhedron, a *CW* complex, a manifold, or admit an action of a topological group. The coefficient group $\mathbb{K}$ can have the additional structure of a module or field, or it can even be replaced by a local coefficient system (**[Whitehead]**) or a presheaf of groups (**[Spanier]**).

It was found very early, in the course of development of obstruction theory, that coefficients of simple type such as groups or rings are not sufficient to describe the lifting problems, and local coefficient systems were introduced by Steenrod in 1942. Both a local coefficient system or a presheaf are functors from some small category induced by a space X to an abelian category.

In order to study an action of a topological group G on a space X, one must consider a large amount of data such as fixed point sets, orbit bundles, and simultaneous representation of quotients of G. Such tasks naturally encouraged the usage of functors and the development of algebra in the functor category. The first chapter, Foundations, in **[tom Dieck]**, is an obvious example of this trend.

The passage from a G-space X to the collection of the fixed point subspaces, $\{X^H$: H is a closed subgroup of G$\}$, motivated W. G. Dwyer and M. Kan to introduce the simplicial model category of functors (diagrams) from a small category $\mathcal{J}$ to simplicial sets **S**, or topological spaces **Top**. The crucial step there was that of taking a functor (diagram), with values in the category of simplicial or topological spaces, in place of a single space X only.

Here we list some simple examples of diagrams of topological spaces (compare **[Farjoun]** 1.8).

(1) Let $\ldots \longrightarrow X_k \longrightarrow X_{k+1} \longrightarrow \ldots$ be a direct system of topological spaces. It is a diagram over the obvious small category.

(2) Let $\mathcal{K}$ be a simplicial complex. Then $\mathcal{K}$ can be thought as a category whose objects are simplices of $\mathcal{K}$ and whose morphisms are inclusions of these simplices. Let $|\mathcal{K}|$ be a geometrical realization of $\mathcal{K}$. Given a topological space X over $|\mathcal{K}|$ with a map $f{:}X \longrightarrow |\mathcal{K}|$ we associate a functor $\underline{X}$ from $\mathcal{K}$ to the category of topological spaces with values $\underline{X}(\sigma) = f^{-1}(|\sigma|)$ for $\sigma \in \mathcal{K}$ and a map $\underline{X}(\sigma) \subseteq \underline{X}(\tau)$ for each inclusion $\sigma < \tau$.

(3) Let $X = \bigcup_{\alpha\in\Lambda} X_\alpha$ be a decomposition of a space X into subspaces. There is a

partial ordering of Λ by the inclusion $X_\sigma \subseteq X_\tau$ for $\sigma,\tau \in \Lambda$. Thus Λ can be regarded as a small category. We may view X as a functor $\underline{X}$ from Λ to the category of topological spaces. If $Y = \bigcup_{\alpha\in\Lambda} Y_\alpha$ and $f: X \longrightarrow Y$ is a map such that $f(X_\gamma) \subseteq Y_\gamma$ for $\gamma \in \Lambda$, then f induces an obvious morphism from the functor $\underline{X}$ to the functor $\underline{Y}$. We may now discuss the existence of a map $g:Y \longrightarrow X$ with $g(Y_\gamma) \subseteq X_\gamma$ such that g is a relative homotopy inverse of f in terms of the functor category $\mathbf{Top}^\Lambda$ and use some type of homology theory for diagrams to find at least partial answers.

(4) Let G be a group and let $L = \{H\}$ be lattice of subgroups $H < G$. This lattice defines a small category with morphisms being the inclusion maps $H \subseteq K$, $H,K \in L$. As a diagram over L one takes the diagram of classifying spaces $\{BH\}_{H\in L}$.

The homotopy theory for diagrams of spaces (and simplicial sets) was developed among others by W. Dwyer, D. Kan, Zabrodsky, E. Dror Farjoun. The development of homology theory for diagrams from the beginning was influenced not only by an already abstract categorical framework but also by a strong tendency, coming within the classical homotopy, to axiomatize homotopy, see **[Quillen]** and **[Baues]**. Here a very illustrative example is that of the work of Shitanda. Shitanda in **[Shitanda]** selected a few axioms for his abstract homotopy category and derived various fundamental theorems expected from a homotopy theory. Next he showed that his axiomatization is closed under the constructions of the functor category and the comma category. Finally, he discussed the case of diagrams taking values in the category of compactly generated Hausdorff spaces and continuous mappings.

As in the classical case, the property of the functor categories depend on the choice of the convenient topological category for values of diagrams. Heller in **[Heller]** and Shitanda showed that a functor category, for the category of compactly generated Hausdorff spaces, allows the use of natural generalizations of notions of fibrations and cofibrations and the computation of all limits and colimits. Another choice for values of diagrams are *k*–spaces as in **[Piacenza 2]**. To secure the existence of all limits and colimits the target category is always assumed to be complete and cocomplete.

Simple examples of diagrams of spaces, in the form of finite commutative scheme of spaces and maps between them, are defined on categories with no extra structure on the hom sets. In full generality, as in **[Shitanda]** or **[Piacenza 2]**, diagrams are defined on a small topological category, taken to be a small category with extra topological structure on hom sets and with continuous composition of morphisms. Enriched category theory, as developed in **[Kelly]** and **[Dubuc]**, gives a very general look at categories with additional structures on hom sets and supplies the enriched version of the Yoneda lemma.

Topological spaces such as *CW* complexes are regarded as very useful because of their relative simplicity, for example, in the classical obstruction theory and homology theory **[Spanier]**. Matumoto developed in detail a theory of equivariant *CW* complexes. In the equivariant theory the standard unit n-ball B^n and the unit (n-1)–sphere S^{n-1} are replaced by the equivariant n–ball $G/H \times B^n$ and (n-1)–sphere $G/H \times S^{n-1}$ respectively. An equivariant *CW* complex is obtained as the colimit of spaces $\{X_n\}$, such that X_n is obtained from X_{n-1} by attaching a collection of equivariant n–cells $\{G/H \times B^n\}$ via the pushout:

$$\begin{array}{ccc} \coprod_{\alpha\in\Lambda_n} G/H_\alpha \underline{x}\, S^{n-1} & \longrightarrow & \coprod_{\alpha\in\Lambda_n} G/H_\alpha \underline{x}\, B^n \\ \phi_n \downarrow & & \downarrow \bar{\phi}_n \\ X_{n-1} & \longrightarrow & X_n \end{array}$$

where each ϕ_n is a G–map (**[Matumoto 1]**, **[tom Dieck]**).

The scheme of equivariant *CW* complexes was easily translated to the realm of functor categories. Each quotient G/H will be replaced by a representable functor (the hom functor) $J(\_, j)$, where j is an object of a small topological category J. Then an n–ball for diagrams has the form $J(\_,j)\times B^n$. A *J-CW* complex (the categorical equivalent of an equivariant *CW* complex) is a diagram X obtained as the colimit of $\{X_n\}$, where each diagram X_n is obtained from X_{n-1} by attaching a collection of n–cells $\{J(\_, j)\times B^n\}$ via the pushout:

$$\begin{array}{ccc} \coprod_{\alpha\in\Lambda_n} J(\_,j_\alpha) \underline{x}\, S^{n-1} & \longrightarrow & \coprod_{\alpha\in\Lambda_n} J(\_,j_\alpha) \underline{x}\, B^n \\ \phi_n \downarrow & & \downarrow \bar{\phi}_n \\ X_{n-1} & \longrightarrow & X_n \end{array}$$

(**[Farjoun]**, **[Piacenza 2]**).

0.2 Uniqueness of Homology Theories

An axiomatic approach to classical homology theory allowed simplification of proofs of theorems since axiomatic proofs are not overburdened with too specific constructions, sometime very tedious, used to define homology groups. In effect, the homology groups of a class of spaces are seen to be completely determined by the coefficient group and this conclusion is formulated in the form of the uniqueness theorem for homology of certain restricted classes of spaces or pairs of spaces.

Eilenberg and Steenrod originally proved the uniqueness theorem for finite triangulated spaces (compact polyhedra). Later the uniqueness theorem was proven for the category of finite *CW* complexes (see **[Hu]** for the proof). If the category satisfies an additional axiom about compact supports, the uniqueness is achieved for a larger class of spaces (see **[Spanier]**, p. 205, for the case of polyhedra).

The axioms for homology of equivariant spaces were formulated in **[Bredon 3]**. They were easily generalized to axioms for homology and cohomology theories for a functor category (**[Dwyer - Kan]**, **[Farjoun]**). The Dimension Axiom requires that the homology group of each representable functor vanishes in nonzero dimensions. A sheaf-type construction of cohomology for diagrams satisfying all the axioms was given by R. Piacenza in **[Piacenza 2]**. Piacenza's sheaf cohomology for diagrams is used in Chapter 4 of this thesis. In **[Shitanda]**, an example of cohomology theory is given for *J-CW* complexes.

In Chapter 2 we first introduce a category of admissible pairs of diagrams, modeled on **[Eilenberg-Steenrod]**. We formulate a set of axioms for homology and cohomology of diagrams and prove the uniqueness theorem for homology and cohomology restricted to the category of finite *J–CW* complexes. We use a version of the Axiom of Compact Supports for diagrams to extend uniqueness to all *J–CW* complexes.

0.3 Singular Homology for Diagrams

In Chapter 3 we present a realization of homology theory for diagrams of topological spaces. The construction given there is modeled on the equivariant singular homology and cohomology proposed by Sören Illman. Illman gave his construction for a G–space which is a topological space with an action of a topological group. It was possible to generalize his method to diagrams of topological spaces. We shall first sketch Illman's construction for homology modules and point out steps which can be easily identified in the method adopted in Chapter 3. The primary reference is **[Illman]**.

Let G be a topological group and let X be a (right) G–space. A family of closed subgroups of G is called an orbit type family for G if $\mathcal{F}$ is closed upon taking conjugate subgroups. $\mathcal{F}$ induces the category J of G–spaces of the form G/H, where H $\in$ $\mathcal{F}$, and G–homotopy classes of G–maps morphisms. Let R be a ring with identity element. A coefficient system M for $\mathcal{F}$ over a ring R is a covariant functor from J to the category of left R-modules. Elements of the module $\mathbb{M}(G/H)$ are called coefficients of the type H. Illman's starting point is a definition of maps of the form:

$$T:\Delta_n \times G/H \longrightarrow X,$$

where Δ_n is the standard n-simplex, $H \in \mathcal{F}$, with a trivial action of G on Δ_n. He calls them **equivariant singular n-simplexes** (of the specific type H) and they play the same role as the classical singular simplexes in ordinary singular homology. A preliminary chain complex is induced by the direct sum of infinite cyclic groups generated by equivariant singular simplexes each tensored with coefficients of the same type:

$$\sum_{T,H\in\mathcal{F}} Z_T \otimes \mathbb{M}(G/H) \quad \text{(T is of the type H).}$$

The boundary operator is induced from the classical one on Δ_n. The next step is to obtain a quotient of the above chain complex with the help of an equivalence relation. The equivalence relation is defined with the help of G-maps of the form

$$h:\Delta_n \times G/H \longrightarrow \Delta_n \times G/K$$

such that the following diagram commutes:

$$\begin{tikzcd} \Delta_n \times G/H \arrow[rr, "h"] \arrow[dr, "pr_1"'] & & \Delta_n \times G/K \arrow[dl, "pr_1"] \\ & \Delta_n & \end{tikzcd},$$

where maps pr_1's are projections on the first factor. Each such map h induces a G-map $h_x:G/H \longrightarrow G/K$ defined by $h_x(gH) = pr_2 \circ h(x,gH)$, where $pr_2:\Delta_n$

$\times$ G/H $\longrightarrow$ G/H is the projection on the second factor and x is any point of Δ_n. Since for $x,y \in \Delta_n$ h_x and h_y are G–homotopic, they induces the same map $\mathbb{M}([h_x]) = \mathbb{M}([h_y]):\mathbb{M}(G/H) \longrightarrow \mathbb{M}(G/K)$. An equivalence relation is defined by

$$T \otimes a \sim T' \otimes a'$$

if there is a G–map h, as above, such that $T = T' \circ h$ and $\mathbb{M}([h_x])(a) = a'$.

Finally, the quotient

$$C_*^G(X;\, \mathbb{M}) = \sum Z_T \otimes \mathbb{M}(G/H) \Big/ \sim$$

is shown to be a chain complex, and the homology of $C_*^G(X;\, \mathbb{M})$ yields Illman's equivariant singular homology of X with coefficients in $\mathbb{M}$, $H_*^G(X;\, \mathbb{M})$. Illman has shown that his homology satisfies all seven equivariant Eilenberg–Steenrod axioms.

In Chapter 3 we give a construction of homology and cohomology modules for diagrams motivated very closely by Illman's exposition. In place of the orbit category $\mathcal{F}$ we use a small topological category J. As was shown in [Vogt 2], a G–space X gives rise to a functor F whose value for each quotient G/H, $H \in \mathcal{F}$, is a topological space F(G/H) equal to a topologized function space of G–maps from G/H to X, or equal to X^H (the fix point set of the subgroup H, see **[tom Dieck]**). Therefore contravariant functors, taking values in some category of topological spaces, are natural generalization of G–spaces. A coefficient system is a homotopy invariant functor on the small topological category taking values in the category of modules.

Now we attempt to find a notion analogous to an equivariant singular n–simplex, that is, to a G–map $T:\Delta_n \times G/H \longrightarrow X$. Since X is a diagram (functor), an arrow denotes a natural transformation. Each G/H can be also easily interpreted as a representable functor $J(\_,a)$, $a \in \text{ob } J$. Therefore we define a natural transformation

$$T:\Delta_n \times J(\_,a) \longrightarrow X$$

$a \in \text{ob } J$, as a new "singular n–simplex" of the type a. However, as we shall show in Chapter 1, an application of version of the Yoneda lemma implies that each such natural transformation corresponds to a continuous map $\Delta_n \longrightarrow X(a)$, which is an ordinary singular n–simplex of the topological space X(a). As we see in **[Vogt 2]** or in **[tom Dieck]**, the classical singular chain complexes of spaces are the ingredients of constructions which lead to the assignment of a homology module to the functor in question.

If we take a closer look at a map $h:\Delta_n \times G/H \longrightarrow \Delta_n \times G/K$, used by Illman to define the equivalence relation, we find that h is uniquely determined by a map of the form $pr_2 \circ h:\Delta_n \times G/H \longrightarrow G/K$. Its functorial analogy has the form of a natural transformation $\Delta_n \times J(\_,a) \longrightarrow J(\_,b)$, a,b $\in$ ob J. Again, the version of the Yoneda lemma 1.1.14(2) reduces any such transformation to a map $\Delta_n \longrightarrow J(a,b)$, which is a singular n–simplex in $J(a,b)$. Again, we observe that singular simplexes of the hom set of the category are used in some, but not all schemes, leading to homology theories for diagrams (compare **[Vogt 2]** and **[tom Dieck]**, or **[Piacenza 2]**).

In Chapter 3 we present the following method of defining homology. Recall that a diagram X is a contravariant functor from a small topological category J to the category of k-spaces. Initially a coefficient system $\mathbb{M}$ is a covariant functor from J to the category of chain complexes of R-modules; later we shall add the assumption that $\mathbb{M}$ takes on values in R-modules only. We proceed in a purely algebraic way, using only tensor products and homs of chain complexes, taking cokernels, and intensively adapting the construction from classical singular homology. The chain complex, whose homology is to be the homology of X with coefficient in $\mathbb{M}$, is defined as the cokernel of the following two chain maps α^X and β^X

$$\sum_{a,b} S_\#(X_b \times J(a,b)) \otimes_{\mathbb{Z}} \mathbb{M}_a \underset{\beta^X}{\overset{\alpha^X}{\rightrightarrows}} \sum_{c} S_\#(X_c) \otimes_{\mathbb{Z}} \mathbb{M}_c,$$

where α is defined by $(\sigma,\eta)\otimes m \longrightarrow \tilde{X}_\#(\sigma,\eta)\otimes m$ and β is defined by $(\sigma,\eta)\otimes m \longrightarrow \sigma\otimes\check{\mathbb{M}}(\eta)(m)$, with $\tilde{X}_\#$ and $\check{\mathbb{M}}$ being induced by X and $\mathbb{M}$ respectively. We prove that the resulting homology theory satisfies all the axioms.

It can be easily seen that there is an obvious similarity of this construction with the equivalence relation defining Illman's chain complex $C^*_G(X, \mathbb{M})$ and with the coend scheme from homological algebra (**[MacLane 1]**, **[tom Dieck]**, **[Vogt 2]**). Since the target chain complex $C^J_*(X, \mathbb{M})$ is defined with the help of chain maps, this method might be likely classified as the coend-type method from enriched category theory (see **[MacLane 1]**, **[Vogt 2]**). However, the other requirements are not satisfied. The maps α^X and β^X are not induced by right and left "modules" (suitable analogies of contravariant and covariant functors), both defined on the same category endowed with the tensor product $\otimes$ (see [Kelly], [Vogt 2], and a notion of the action in [MacLane 2]).

Vogt, in his preprint [Vogt 2], applied a singular functor to J and obtained a differential graded category. In order to produce a right "module" from X Vogt composed the Eilenberg-Zilber chain equivalence map $\varepsilon: S_{\#}(Z)\otimes S_{\#}(W) \longrightarrow S_{\#}(Z\times W)$, where Z and W are topological spaces, with maps induced by X and the singular functor (see $\tilde{X}_{ab\#}: S_{\#}(X_b \times J(a,b)) \longrightarrow S_{\#}(X_a)$ described in Chapter 3). A different approach is chosen in Chapter 3. The method of Chapter 3 uses a simple property of the simplicial structure of the the singular chain complex of the product Z×W; it allows a representation of a singular n-simplex of Z×W as a pair of two n-simplexes, one of Z and the other of W (see 3.1.5).

There is another difference between the generality of the approach in [Vogt 2] and that in Chapter 3. When showing that the Excision Axiom is satisfied, Vogt's arguments are explicitly based on the assumption that the coefficient system takes its values in R-modules, and not in chain complexes. In this thesis Excision is proved without such restriction, for general coefficient systems taking values in the category of chain complexes. The restriction on the type of coefficients is used only to prove that the proposed homology satisfies the (ordinary) Dimension Axiom.

0.4 Comparison Theorem

Here we explain the origin of the Comparison Theorem (4.2.20). Let a topological space X be 0-connected and semilocally 1-connected. Eilenberg's classical theorem (**[Whitehead]**, Chapter VI, Theorem 3.4*) establishes the isomorphism between the cohomology of a space X with the local system of coefficients and the equivariant cohomology of the universal covering of X.

In **[Piacenza 1]** this result of Eilenberg was generalized by replacing a universal covering space by a G–space and a left G–module by a coefficient system $\mathbb{K}$. Theorem 3.11 in **[Piacenza 1]** establishes the isomorphism between Illman's cohomology and a sheaf cohomology. The statement of Piacenza's Theorem 3.11 will follow after the necessary definitions.

Let G be a topological group and let X be a (right) G–space with its natural quotient $q:X \longrightarrow X/G$. Let $\mathcal{F}$ be an orbit family for G as in 0.3 and let $\mathcal{J}$ be the full subcategory of the category of right G–spaces with objects G/H for $H \in \mathcal{F}$. Let $\mathbb{K}$ denote a homotopy invariant contravariant coefficient system on $\mathcal{J}$, that is, $\mathbb{K}$ is a contravariant functor from $\mathcal{J}$ to **Ab** such that $\mathbb{K}(f) = \mathbb{K}(g)$ whenever $f,g:G/H \longrightarrow G/K$ are homotopic by an equivariant homotopy (the same as the original Illman assumption which defines $\mathbb{K}$ on the homotopy category of $\mathcal{J}$). By $C^n_G(X; \mathbb{K})$ we denote the group of equivariant singular cochains as defined in **[Illman]**. The cochain complex $C^*_G(X; \mathbb{K})$ gives us the singular equivariant cohomology groups $H^n_G(X; \mathbb{K})$ (see **[Illman]**). Let S^* be a cochain of presheaves over X/G, defined by setting $S^n(U) = C^n_G(q^{-1}(U);\mathbb{K})$, for U open in X/G, and let $\tilde{\mathbb{K}}$ be a sheaf obtained by taking the augmentation of S^*. A G–space X is said to be G cohomologically connected (G–*clc*) if the complex of sheaves $0 \longrightarrow \tilde{\mathbb{K}} \longrightarrow S^*$ is exact for any homotopy invariant system $\mathbb{K}$.

Theorem ([Piacenza 1], 3.11). If a G–space X is G–*clc* with X/G paracompact then there is a natural isomorphism:

$$H^*_G(X; \mathbb{K}) \cong H^*(X/G; \tilde{\mathbb{K}}),$$

where the right side is sheaf cohomology and $\mathbb{K}$ is any homotopy invariant coefficient system.

In Chapter 4 we prove the following analogous theorem for diagrams of topological spaces.

The Comparison Theorem (4.2.20). If $X \in$ ob J–CGV is J–*clc*, with all X_a, $J(a,b)$, $a,b \in$ ob J, locally path connected, and with X/J paracompact, then there is a natural isomorphism:

$$H^*_J(X; \mathbb{K}) \cong {}_{sh}H^*_J(X; \mathbb{K}).$$

between the singular and sheaf J–cohomologies of X with coefficients in $\mathbb{K}$.

Here the orbit category $\mathcal{F}$ of G is replaced by any small topological category (1.1.4). The G–space is replaced by a diagram of topological spaces on J (1.1.6). The cohomology $H^*_J(X; \mathbb{K})$ is the singular J–cohomology derived in Chapter 3. The sheaf J–cohomology ${}_{sh}H^*_J(X; \mathbb{K})$ is then by definition the sheaf cohomology $H^*(X/J; \mathcal{K}X)$ of the colimit space X/J with coefficients in a sheaf $\mathcal{K}X$ (4.1.4). The sheaf $\mathcal{K}X$ is derived analogously to $\tilde{\kappa}$ (see 4.2.12), and the condition *J–clc* is phrased similarly to the *G cls* condition (see 4.2.17).

0.5 Overview of Sections

(1.1) We describe the category of *k*–spaces and give a list of their properties ([Vogt 1]). We define diagrams of topological spaces as continuous functors on a small topological category with values in *k*–spaces and list the property of the resulting functor category.

(1.2) We present here homotopy theory for diagrams. Definitions and listed properties are taken from [Piacenza 2] and [Heller].

(1.3) In this section we give the definition of homotopy groups for diagrams following [Piacenza 2].

(1.4) A construction of cellular structures (*J–CW* complexes) for diagrams is presented. Their properties, paralleling that of the classical *CW* complexes, are listed. The standard references is [Piacenza 2].

(1.5) This section contains facts from [Piacenza 2] on Quillen's closed model category structure for diagrams.

(1.6) In this section we develop the collaring technique for diagrams along the lines of **[Fritsch - Piccinini]**. We apply collars in a proof of Piacenza's Local Contractibility Theorem stated in the sections 1.4.

(2.1) The section contains the definition of admissible pairs of diagrams following **[Eilenberg - Steenrod]**. Examples of these are given.

(2.2) & (2.3) We formulate system of axioms for homology and cohomology for diagrams of topological spaces, paralleling the classical formulation from **[Eilenberg - Steenrod]**.

(2.4) & (2.5) In these sections we prove that homology and cohomology theories, when restricted to finite *J–CW* complexes, are determined by their coefficient systems. We follow the axiomatic approach in **[Hu]**. We use a version of the Axiom of Compact Supports for diagrams to extend uniqueness to arbitrary *J–CW* complexes.

(2.6) We derive the Mayer-Vietoris sequence for diagrams.

(3.1) We give our notation for chain and cochain complexes of modules and their tensor and hom products.

(3.2) Coefficients systems for homology and cohomology of diagrams are defined in full generality as functors taking values in the category of chain complexes of modules. Extensions of coefficients are defined following **[Illman]**.

(3.3) & (3.4) In these sections we apply the classical singular functor pointwise to a diagram and to the small topological category. We use cokernels of chain maps to define a singular chain of the diagram in a process which resembles taking a tensor product (a specific coend) over a category ([tom Dieck], I.11.6). We use kernels of cochain maps to obtain a singular cochain of the diagram in a process similar to creating the hom product (a specific end) over a category ([Vogt 2]). We prove that both singular homology and cohomology for diagrams satisfy the Excision Axiom.

(3.5) & (3.6) Properties of the classical singular functor allow us to obtain the induced morphisms between singular chains, and cochains respectively.

(3.7) & (3.8) In both sections we first discuss the case of a homotopy map on a topological space and obtain explicit (co)chain homotopy maps of singular (co)chain complexes, all induced by the classical singular functor. Next, given a homotopy between morphisms of diagrams, we construct a pair of commuting homotopies and obtain that the induced map between the appropriate cokernels (or kernels) is a (co)chain homotopy map between induced morphisms. That way we prove that both singular homology and cohomology for diagrams satisfy the Homotopy Axiom.

(3.9) & (3.10) First we prove that both singular homology and cohomology for diagrams satisfy the Excision Axiom for "coverings", paralleling the property of the classical singular (co)homology for topological spaces for (open) coverings. We show that analogies for diagrams are induced by the classical subdivision functor and its homotopy. We prove that we can correctly define a singular sub(co)chain induced by a "covering" of a diagram. As in **[Vogt 2]** we use the same classical scheme to define the subdivision map and its homotopy, both subordinated to the given "covering". The reference is **[Spanier]**.

(3.11) & (3.12) We obtain an explicit formula for the singular chain complex of a representable functor as the tensor product of the (classical) singular complex of a one point space and the appropriate value of a coefficient system. Similarly for cohomology, the singular cochain complex of a representable functor is the hom product of the (classical) singular complex of a one point space and the appropriate value of the coefficient system. If we choose a coefficient system taking values in modules or abelian groups only, then the Dimension Axiom is satisfied.

(3.13) In this section we apply the additive properties of the classical singular functor with respect to a disjoint union of spaces to prove that both singular homology and cohomology satisfy the Additivity Axiom.

(3.14) In this section we prove that in the case of a discrete category J the singular J-homology and J-cohomology of any diagram are given by the classical coend and end construction (**[MacLane 2]** and **[Piacenza 1]**).

(4.1) For a given diagram and a coefficient system, a presheaf of modules is defined on the colimit space of the diagram as in **[Piacenza 2]**. A direct application of the classical sheaf theory allows us to obtain a sheaf cohomology for diagrams satisfying all the axioms. The standard references is **[Bredon 1]**.

(4.2) We prove the comparison theorem between singular cohomology for diagrams and sheaf cohomology for diagrams. The method of the proof using acyclic resolutions of sheaves follows along the lines of **[Piacenza 1]**.

CHAPTER 1

HOMOTOPY THEORY AND CELLULAR THEORY OF DIAGRAMS

In this chapter we gather together fundamental facts and definitions, most of which are of a topological and a categorical nature. The category of k–spaces **CGV** is introduced following Vogt's approach. The list of properties of **CGV** is presented, such as the existence of the categorical product and the exponential law. The **CGV** category is a Cartesian closed category and moreover it is complete and cocomplete. The **CGV** category plays the role of our convenient topological category. A small topological category is introduced as a small category enriched in **CGV**. The contravariant continuous functors on a small topological category taking values in **CGV** are called diagrams of topological spaces. The category of such diagrams has many properties of **CGV** as completeness and cocompleteness, and it is enriched in **CGV** by an enriched version of the Yoneda lemma. It admits the development of homotopy theory which is presented in Section 1.2 with the concentration on cofibrations. Homotopy groups for diagrams are presented in Section 1.3 following Piacenza's formulation in **[Piacenza 2]**. Cellular constructions are presented in Section 1.4 with major results paralleling classical ones taken from **[Piacenza 2]**. In the cellular theory for diagrams, cells are defined as the product of representable functors and ordinary n–balls. This approach generalized the classical and the equivariant CW–complexes. Section 1.5 contains facts from **[Piacenza 2]** on the Quillen's closed model structure for diagrams. In Section 1.6 the collaring technique is used to prove a few lemmas.

1.1 Definition of the Basic Category: *k*-spaces

Let **Top** denote the category of topological spaces, **CHaus** the category of compact Hausdorff spaces, and **LocCHaus** the category of locally compact spaces. Here we define our basic category **CGV** as the category of compactly generated spaces in the sense of Vogt. The construction of **CGV** is carried out in a few steps following the more general construction of [**VOGT 1**] p.546.

1.1.1 Definition. Let X be a topological space. Denote by **CHaus/X** the category whose objects are all maps $f:B_f \longrightarrow X$ in Top, where $B_f \in$ *ob* CHaus, and whose morphisms from f to g are all maps $h:B_f \longrightarrow B_g$ such that $f = g \circ h$. The spaces B_f, $f \in$ *ob* **CHaus/X**, and the maps $h:B_f \longrightarrow B_g$ form a system $D(X)$ in **Top**. Define $k(X)$ = **colim** $D(X)$. Then **CGV** is the full subcategory of Top consisting of all objects $k(X)$ with $X \in$ *ob* Top.

We use the canonical representation of $k(X)$ with the same underlining set as X but endowed with a finer topology induced by the colimit in Top ([**Vogt 1**] 1.1 p 546). This new topology on X will be called the **Vogt compactly generated topology** on X or the *k*-topology on X ([**Brown**] Section 5.9). A (set) map $f:X \longrightarrow Y$ between topological spaces X and Y is said to be a map in **CGV** if f is a morphism of **CGV**-spaces $k(X)$ and $k(Y)$.

Clearly a topological space X is an object of **CGV**(or **CGV**-space) if the topology on X is determined by all possible (continuous) maps from compact Hausdorff spaces to X. The category **CGV** differs from the category of compactly generated spaces in the sense of Steenrod (denoted by **CGHaus** in **[MacLane 2]** p. 181 and by **CGH** in **[Shitanda]** p. 481), or from May's compactly generated spaces, **[May 1]** p. III-4, by lack of any separation condition on the topology. **CGV** is denoted by $\mathscr{K}\mathscr{G}$ in **[Vogt 1]** p. 554.

1.1.2 Properties of CGV. Facts:

(1) **CGV** is complete and cocomplete (**[Vogt 1]** 5.1(a) & 2.1(d)).

(2) The forgetful functor **CGV** $\longrightarrow$ **Set** preserves limits and colimits (**[Vogt 1]** 5.1(a) & 2.1(d)).

(2) A quotient space of a space in **CGV** is in **CGV** (**[Vogt 1]** 5.1(a) & 2.2).

(3) An open subspace of a space in **CGV** is in **CGV** (**[Vogt 1]** 5.1(a) & 2.4).

(4) A closed subspace of a space in **CGV** is in **CGV** (**[Vogt 1]** 5.1(a) & 2.4).

(5) The internal hom space **CGV**(X,Y) can be regarded canonically as an object of **CGV** (**[Vogt 1]** 5.1(a) & the definition on p. 550).

(6) If the product in **Top** is denoted by $\times$, and the product in **CGV** is denoted by $\underline{x}$, then $k(X\times Y) = X\underline{x}Y$ (**[Vogt 1]** 5.1(a) & the definition on p. 549).

(7) If X is a **LocCHaus**-space and Y is a **CGV**-space, then $X\times Y = X\underline{x}Y$ (**[Vogt 1]** 5.1(c)).

(8) **CGV**($X\underline{x}Y$, Z) $\cong$ **CGV**(X, **CGV**(Y, Z)) and **CGV** is a Cartesian closed category (**[Vogt 1]** 5.1(a) & 3.6).

(9) $\varinjlim (X_j\underline{x}Y) = (\varinjlim X_j)\underline{x}Y$ (**[Vogt 1]** 5.1(a) & 3.7(b)).

(10) Let $f:X \longrightarrow X'$ and $g:Y \longrightarrow Y'$ be identification maps between spaces in **CGV**. Then $f\underline{x}g:X\underline{x}Y \longrightarrow X'\underline{x}Y'$ is an identification map ([**Vogt 1**] 5.1(a) & 3.8).

(11) The identity map $k(X) \longrightarrow X$, $X \in$ *ob* **Top**, induces isomorphism of homotopy and homology and cohomology groups ([**Vogt 1**] 5.1(d)).

(12) Let $\mathbf{D}$ be any diagram in **GCV** (it may be big). If the colimit $C = \varinjlim \mathbf{D}$ exists in **Top**, then it exists in **CGV** and equals C ([**Vogt 1**] 2.1(a)). If the limit $L = \varprojlim \mathbf{D}$ exists in **Top**, then it exists in **CGV** and equals $k(L)$ ([**Vogt 1**] 2.1(b)).

1.1.3 Definition A **pair** (X, A) of **CGV**-spaces consists of two spaces X and A in **CGV** together with some embedding $t:A \longrightarrow X$ called the **inclusion map** and also denoted by $A \subset X$. The category of all pairs in **CGV** and their maps is denoted by $\mathbf{CGV}^2$.

1.1.4 Definition (Topological Category). A category J is called a **topological category** if each set of morphisms $J(a,b)$ is a **CGV**-space, the associative composition $m_J:J(b,c)\times J(a,b) \longrightarrow J(a,c)$ is continuous in the k-topology for any $a,b,c \in$ *ob* J, and for each $a \in$ ob J there is an **identity element** $id_a \in J(a,a)$ such that $m_J(id_b,x) = m_J(x,id_a) = x$ for $x \in J(a,b)$.

Note that a topological category J is a **CGV**-category in the sense of enriched category theory ([**Dubuc**], [**Kelly**]).

1.1.5 Examples

(1) Any concrete category can always be regarded as a topological category if the internal hom sets are endowed with the discrete topology.

(2) The category **CGV** is a topological category by 1.1.2(5) & **[Kelly]** p.36.

(3) A **compactly generated group** G in **CGV** (e.g. group) is a topological group such that both a composition function $G \times G \longrightarrow G$ and the inverse function $G \longrightarrow G$ are continuous in the k-topology. Such space is treated as a topological category with one object $*$ and with all morphisms invertible where $G = G(*,*)$ is in **CGV**.

(4) **The orbit category** $Or = Or(\mathrm{G}, \mathcal{F})$ (**[tom Dieck]** p. 72). Let G be a compact Lie group and $\mathcal{F}$ be any family of closed subgroups of G which are closed under conjugation. If X and Y are G–spaces we denote by $C_G(X,Y)$ a space of the G–maps from X to Y with the compact–open topology. The objects of Or are the homogeneous spaces G/H for $H \in \mathcal{F}$, and $Or(G/H, G/K) = C_G(G/H, G/K)$. Since $Or(G/H, G/K) \cong (G/K)^H$ is a compact set (**[tom Dieck]** 3.8), Or becomes a small topological category.

1.1.6 Definition (Continuous Functors). A contravariant functor $F{:}J \longrightarrow C$ between topological categories is **continuous** if the induced map $F_{a,b}{:}J(a,b) \longrightarrow C(F(a),F(b))$ is continuous for any objects a, b in J.

Note that 1.1.6 defines in fact a **CGV**-functor between **CGV**-categories.

1.1.7 Definition (Functor Category and Diagrams of Topological Spaces). Let J be a small topological category. Define a category J-**CGV** of diagrams over J as the functor category:

J-**CGV** = *Continuous Contravariant Functors from J to* **CGV**,

whose objects, called J-**CGV**-spaces or topological diagrams over J, are continuous contravariant functors $X{:}J \longrightarrow$ **CGV** and whose morphisms, called J-**CGV**-morphisms, are (continuous) natural transformations between them (denoted by *J-Top* in **[Piacenza 2]** or by $\mathcal{F}$ in **[Shitanda]** for **CGH**-spaces).

We denote by $\mathbf{CGV}^J$ the (ordinary) functor category of (all) functors from J to **CGV** and their natural transformations. If necessary we refer to objects in J-**CGV** as *continuous* diagrams as opposed to those in $\mathbf{CGV}^{J^{op}}$. Similarly, we define J-**Ab** or J-$_R$**Mod**, the categories of continuous contravariant functors from J to the category of abelian groups **Ab** or (left) R-modules $_R$**Mod**.

1.1.8 Examples

(1) Let $J = \{*\}$ be the category with one object and one morphism. Then any space in **CGV** may be regarded as a functor from J to **CGV** and J-**CGV** $\cong$ **CGV**.

(2) Let $J = (*_1 \longrightarrow *_2)$ be the category with two objects and one morphism between them. Then any map $f{:}X \longrightarrow Y$ in **CGV** can be regard as a functor from J to **CGV**.

(3) Every object j in J defines the continuous contravariant functor $D_j = J(\_,j)\colon J \longrightarrow$ **CGV** called the J-orbit at j (see **[Farjoun]** p. 101).

(4) Let G be a compact group. Then G is a c.g. group (see 1.1.5(3)). The category of (right) G-spaces in **CGV** is, in fact, the category G-**CGV**. The adjunction map in **CGV** allows us to treat each right G-space as a continuous contravariant functor from the c.g. group G to **CGV**.

(5) Any **CGV**-space X can be regarded as a diagram $\Delta(X)$ over J by $\Delta(X)(j) = X$, $\Delta(X)(f) = Id_X$ for any $j \in ob\ J$ and $f \in morph\ J$, and $\Delta(f)(j) = f$ for $f:X \longrightarrow Y$.

(6) Let $Or(G, \mathcal{F})$ be an orbit category described in Example 1.1.5(4). Every G–space X in **CGV** induces a J–diagram $X:Or^{op} \longrightarrow$ **CGV** given by $X(G/H) = C_G(G/H, X)$. By **[tom Dieck]**, 3.8, $X(G/H)$ is homeomorphic to X^H, a closed subspace of points of X fixed by H. Continuity of $X_{H,K}$ follows from that of the adjoint map $\tilde{X}_{H,K}:X(G/K)\underline{x}Or(G/H, G/K) \longrightarrow X(G/H)$, $H,K \in \mathcal{F}$, which is easily seen to be the composition function (**[Vogt 2]**)
$C_G(G/K, X)\underline{x}C_G(G/H, G/K) \longrightarrow C_G(G/H, X)$.

We shall see that J-**CGV** has products, pullbacks, and pushouts just as the Vogt category of compactly generated spaces **CGV** does.

1.1.9 Fact. The category J-**CGV** is complete and cocomplete.

Proof. Repetition of Shitanda's proof, **[Shitanda]** 3.2 p. 489, after changing his **CGH** to **CGV**.

The category J-**CGV** is naturally enriched in **CGV**; that is, we have:

1.1.10 Fact. The category of diagrams J-**CGV** is a **CGV**-category and

$$J\text{-}\mathbf{CGV}(X,Y) = \int_J \mathbf{CGV}(Xj,Yj)$$

for any diagrams X, Y.

Proof. Since J is small, the enriched end $\int_J \mathbf{CGV}(Xj,Yj)$ exists, belongs to

CGV, and its underlining set is the class J-CGV(X,Y) of all natural transformations from X to Y, by **[Dubuc]**, p. 57-58, (1).

1.1.11 Fact (The Strong Yoneda Lemma for diagrams). Let F be a diagram over J and $r \in ob\ J$. Then there is an isomorphism of CGV-spaces

$$J\text{-CGV}\left(J(\_,r),\ F\_\right) = \int_j \text{CGV}\left(J(j,r),\ Fj\right) \cong Fr.$$

Proof. This is implied by **[Kelly]** p. 65, (2.31) for V = CGV and $A = J$.

1.1.12 Definition. Let $X \in ob\ J$-CGV and $C \in ob$ CGV. Define a **product** (a **tensor product**) of X with a space C as a diagram $X\underline{x}C$ by $(X\underline{x}C)(j) = Xj\underline{x}C$, $j \in ob\ J$ and $(X\underline{x}C)(f) = Xf\underline{x}id_C$, $f \in morph\ J$. Moreover if $F{:}X \longrightarrow Y$ in J-CGV and $g{:}C \longrightarrow D$ in CGV, define $F\underline{x}g{:}X\underline{x}C \longrightarrow Y\underline{x}D$ by $(F\underline{x}g)(j) = Fj\underline{x}g$, $j \in ob\ J$.

1.1.13 Definition. Let $X \in ob\ J$-CGV and $C \in ob$ CGV. Define a **power** of X with a space C (a **cotensor product** of X with a space C) as a diagram X^C by $(X^C)(j) = \text{CGV}(C,Xj)$, $j \in ob\ J$ and $(X^C)(f) = \text{CGV}(\_,Xf)$, $f \in morph\ J$. Moreover if $F{:}X \longrightarrow Y$ in J-CGV and $g{:}C \longrightarrow D$ in CGV define $F^g{:}X^D \longrightarrow Y^C$ by $(F^g)(j) = \text{CGV}(\_,\ Fj{\circ}g)$, $j \in ob\ J$.

1.1.14 Fact. Let X,Y be two objects of J-CGV, C an object of J-CGV, and $j \in ob\ J$. Then:

(1) J-CGV$(X\underline{x}C,\ Y) \cong J$-CGV$(X,\ Y^C)$ (**[Farjoun]** p. 96 with hom(C,Y) = Y^C)

(2) J-CGV$(D_j\underline{x}C,\ X) \cong$ CGV$(C,\ Xj)$ (**[Shitanda]** Lemma 3.4, p.490 or apply 1.1.11 & 1.1.14(1)).

1.1.15 A morphism f:X⟶Y of J-**CGV** is called an **embedding** if for each $j \in$ ob J the map $fj:Xj \longrightarrow Yj$ is an embedding. A J-**pair** (X, A) consists of two objects X and A of J-**CGV** together with some, usually unnamed, embedding $t:A \longrightarrow X$. We call t the **inclusion** morphism and write $A \subset X$. Such an object A is called a **subdiagram** of X. **A morphism** f **of** J-**pairs** of diagrams, $f:(X,A) \longrightarrow (X',A')$, is defined by the obvious commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle t}\uparrow & & \uparrow{\scriptstyle t'} \\ A & \xrightarrow{f'} & A' \end{array}$$

where t and t′ are embeddings which define pairs, and f′ is usually unnamed and denoted $f|Y$. These definitions are extended to (J–) **subpairs, triples** or **n-tuples, n-ads,** and their morphisms in the standard way. The category of all J-pairs and their morphisms is denoted by J-$\mathbf{CGV}^2$. A morphism $f:X \longrightarrow Y$ of J-**CGV** is called **closed** (resp. **open**) if for each $j \in$ ob J, the function fj is closed (resp. open); analogously we define a closed or open J-pair (X,A) and A is called a closed (resp. open) subdiagram of X.

1.2 Homotopy Theory

One possible way to obtain most results for homotopy theory of diagrams without direct recalculations in the category J-**CGV** is to use the axiomatic approach as in **[Baues]** and **[Shitanda]**. Both **[Baues]** and **[Shitanda]** introduced sets of axioms, based on the cylinder and path functors, and using these they defined abstract homotopy categories. From these axioms they developed various properties and theorems, including those one might expect from homotopy theory. Shitanda showed in his paper **[Shitanda]** that the functor category C^D (for a small category D and his abstract homotopy category C) is again an abstract homotopy category, inducing all theorems and properties now true in the functor category and its "continuous" part D-C. Shitanda's final applications to **CGH** valued diagrams can be easily changed to **CGV**.

We content ourself with the following list of useful definitions and facts.

1.2.1 A **homotopy** in category J-**CGV** is a morphism $X\underline{x}I \longrightarrow Y$ of J-**CGV**, where X and Y are diagrams over J. The resulting equivalence relation on the morphisms of J-**CGV** gives rise to the **quotient homotopy category** hJ-**CGV**. [f] denotes the **homotopy class** of f. An isomorphism in hJ-**CGV** is called a J-**homotopy equivalence.**

1.2.2 A morphism of J-**CGV** is called a J-**cofibration** if it has the homotopy extension property (HEP) with respect to any diagram over J. A morphism of J-**CGV** is called a J-**fibration** if it has the homotopy lifting property (HLP) with respect to any diagram over J. See **[Brown]** for standard notation on cofibrations.

1.2.3 Lemma (The existence of cofibrations in J-CGV). Let $r \in \text{ob } J$. If $u{:}A \longrightarrow X$ is a closed cofibration between **LocCHaus**-spaces, then

$$\mathrm{id}_{D_r} \underline{x}\; u{:}D_r \,\underline{x}\, A \longrightarrow D_r \,\underline{x}\, X$$

is a closed J-cofibration such that for each $j \in \text{ob } J$ the map $\left(\mathrm{id}_{D_r} \underline{x}\; u\right)(j)$

$= \mathrm{id}_{J(j,r)} \underline{\times}\, u{:}J(r,j) \underline{\times} A \longrightarrow J(r,j) \underline{\times} X$ is a closed cofibration in **CGV**.

Proof. Let the following diagram (on the left) describe the lifting homotopy problem in J–**CGV** (see **[Spanier]** p. 66).

$$\begin{tikzcd} D_r\underline{\times}A\underline{\times}\{0\} \arrow[rr] \arrow[dd] & & D_r\underline{\times}X\underline{\times}\{0\} \arrow[dl, "g"'] \arrow[dd] \\ & Y & \\ D_r\underline{\times}A\underline{\times}I \arrow[rr] \arrow[ur, "G"] & & D_r\underline{\times}X\underline{\times}I \arrow[ul, dashed, "F"'] \end{tikzcd} \qquad \begin{tikzcd} A\underline{\times}\{0\} \arrow[rr] \arrow[dd] & & X\underline{\times}\{0\} \arrow[dl, "g_r"'] \arrow[dd] \\ & Y_r & \\ A\underline{\times}I \arrow[rr] \arrow[ur, "G_r"] & & X\underline{\times}I \arrow[ul, dashed, "F_r"'] \end{tikzcd}$$

We apply the Yoneda lemma in the version of 1.1.14(2) and get the extension problem in **CGV** (the diagram on the right). After finding F_r, we apply 1.1.14(2) again and obtain the required F.

1.2.4 Corollary. Let $r \in$ ob J. Using cofibrations $S^{n-1} \overset{u}{\subset} B^n$ and $\{0\} \overset{v}{\subset} I^n$ we find that $D_r\underline{\times}S^{n-1} \xrightarrow{id \times u} D_r\underline{\times}B^n$ and $D_r\underline{\times}\{0\} \xrightarrow{id \times v} D_r\underline{\times}I^n$ are closed J-cofibrations.

1.2.5 Facts. Here are a few facts on cofibrations in **Top** and **CGV**.

(1) Let $u{:}A \longrightarrow X$ be a cofibration in **Top**. Then u is an embedding (**[Brown]** 7.2.4(i), p.251).

(2) Let A and X be spaces in **CGV** and $u{:}A \longrightarrow X$ be a closed cofibration in **Top**. Then u is a closed cofibration and an embedding in **CGV** ((1), 1.1.2(7), 1.1.2(4)).

(3) Let u be a cofibration map in **CGV**. Then pushouts $\bar{u}$ of u along any map from **CGV** taken in the categories **CGV** and **Top** are the same. Moreover, if u

is a closed cofibration, then $\bar{u}$ is a closed cofibration too (**[Vogt 1]** 2.1(b) p.548 and **[Brown]** 4.5.5(a) for the second part).

1.2.6 Facts ([Heller] p.188-189).

(1) Any morphism of diagrams over J can be factored as a closed J-cofibration followed by a J-homotopy equivalence and as a J-homotopy equivalence followed by a J-fibration.

(2) A composition of J-cofibrations (J-fibrations) is a J-cofibration (J-fibration).

(3) Coproducts of J-cofibrations are J-cofibrations.

(4) Products of J-fibrations are J-fibrations.

(5) Both products and coproducts of J-homotopy equivalences are J-homotopy equivalences.

(6) A pushout of a J-cofibration along any morphism is again a J-cofibration.

(7) A pullback of a J-fibration along any morphism is again a J-fibration.

1.2.7 Fact (Invariance of Pushouts) (**[Piacenza 2]** p. 815). Suppose that we have a commutative diagram

$$\begin{tikzcd}
 & X' \arrow[rr] & & Z' \\
X \arrow[ur, "\alpha"] \arrow[rr] & & Z \arrow[ur, "\theta"] & \\
 & A' \arrow[uu, "i'"] \arrow[rr, "f'"] & & Y' \arrow[uu] \\
A \arrow[uu, "i"] \arrow[ur, "\beta"] \arrow[rr, "f"'] & & Y \arrow[uu] \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

in which i and i' are closed J-cofibrations, f and f' are arbitrary morphisms in J-**CGV**, α, β and γ are J-homotopy equivalences, and the front and back faces are pushouts. Then θ, the induced map on the pushouts, is also a J-homotopy equivalence.

1.2.8 Facts ([Heller] p.188-189).

(1) A pushout of a closed J-cofibration which is also a J-homotopy equivalence is again a J-homotopy equivalence.

(2) A pullback of a J-fibration which is also a J-homotopy equivalence is again a J-homotopy equivalence.

(3) A pushout of a J-homotopy equivalence along a closed J-cofibration is a J-homotopy equivalence.

(4) A pullback of a J-homotopy equivalence along a J-fibration is a J-homotopy equivalence.

1.2.9 Fact (Invariance of Colimits over Cofibrations) ([Piacenza 2] p. 815). Suppose given a J-homotopy commutative diagram

$$\begin{tikzcd} X^0 \arrow[r, "\alpha_0"] \arrow[d, "f^0"] & X^1 \arrow[r, "\alpha_1"] \arrow[d, "f^1"] & \cdots \arrow[r] & X^k \arrow[r, "\alpha_k"] \arrow[d, "f^k"] & \cdots \\ Y^0 \arrow[r, "\beta_0"] & Y^1 \arrow[r, "\beta_1"] & \cdots \arrow[r] & Y^k \arrow[r, "\beta_k"] & \cdots \end{tikzcd}$$

in J-**CGV** where the α_k and β_k are J-cofibrations and f^k are J-homotopy equivalences. Then the map $colim_k f^k$: $colim_k X^k \longrightarrow colim_k Y^k$, which is defined up to homotopy, is a J-homotopy equivalence.

1.3. Homotopy Groups of Diagrams

1.3.1 In the attempt to define the homotopy groups for diagrams the primary obstacle is the requirement of a base point in a diagram which is a very strong assumption. Following Piacenza, the usage of the Yoneda lemma allows us to interpret the homotopy groups of Xj with a specific $x_0 \in Xj$ in terms of diagrams (see **[Piacenza 2]** p. 816).

1.3.2 Let X be a diagram over J, $j \in J$, and $x_0 \in Xj$. Denote by $c_0:\partial I^n \longrightarrow X_j$ the unique map with image $\{x_0\}$. Note that the homotopy class of a map $f:(I^n,\partial I^n) \longrightarrow (Xj,\{x_0\})$ in **CGV** with homotopy relative to $\{\partial I^n;\ \{x_0\}\}$ corresponds uniquely to the homotopy class of a map $f:(I^n,\partial I^n) \longrightarrow (Xj,Xj)$ with $f|\partial I^n = c_0$ and with homotopy maps restricting to c_0 on $I\underline{x}\partial I^n$. By 1.12(b) the map c_0 corresponds to some $\varphi_{x.}:D_j\ \underline{x}\ \partial I^n \longrightarrow X$ in J-**CGV** and f corresponds to $\bar{f}:D_j\ \underline{x}\ (I^n,\partial I^n) \longrightarrow (X,X)$ with $\bar{f}|D_j\underline{x}\partial I^n = \varphi_{x.}$ and with homotopy morphisms restricting to $\varphi_{x.}$ on $D_j\underline{x}\partial I^n$. The ***j*–n^th^ homotopy group** of X at $= x_0 \in X(j)$ is defined by

$$\pi_n^j(X,\ \varphi_{x.}) = \left\{[\bar{f}] \in hJ\text{-}\mathbf{CGV}\left[D_j\underline{x}(I^n,\partial I^n),(X,X)\right] \mid \bar{f}|D_j\underline{x}\partial I^n = \varphi_{x.}\right\}.$$

1.3.3 Let (X,Y) be a pair in J-**CGV**, $j \in J$, and $y_0 \in Yj$. Denote by $c_0:\{0\} \longrightarrow Y_j$ the unique map with image $\{y_0\}$. Note that the homotopy class of a map $f:(I^n,\partial I^n,\{0\}) \longrightarrow (Xj,Yj,\{y_0\})$ in **CGV** with homotopy relative to $\{(\partial I^n,\ \{0\});\ (Yj,\{y_0\})\}$ corresponds uniquely to the homotopy class of a map $f:(I^n,\partial I^n,\{0\}) \longrightarrow (Xj,Yj,Yj)$ with $f|\{0\} = c_0$ and with homotopy maps restricting to c_0 on $I\underline{x}\{0\}$. By 1.12(b) the map c_0 corresponds to some $\varphi_{y_\cdot}: D_j\underline{x}\{0\} \longrightarrow Y$ in J-CGV, and f corresponds to $\bar{f}: D_j\underline{x}(I^n, \partial I^n, \{0\}) \longrightarrow (X,Y,Y)$ with $\bar{f}|D_j\underline{x}\{0\} = \varphi_{y_\cdot}$ and with homotopy morphisms restricting to $\varphi_{y_\cdot}$ on $D_j\underline{x}\{0\}$. The **relative j-nth homotopy group** of X at $y_0 \in Y(j)$ is defined by

$$\pi_n^j(X,Y,\varphi_{y_\cdot}) = \left\{[\bar{f}] \in hJ\text{-CGV}\left[D_j\underline{x}(I^n,\partial I^n,\{0\}),\ (X,Y,Y)\right] \mid \ \bar{f}|D_j\underline{x}\{0\} = \varphi_{y_\cdot}\right\}.$$

1.3.4 Fact ([Piacenza 2] 2.2 p. 816). Let $j \in$ ob J. There are natural equivalences $\pi_n^j(X, \varphi_{x_\cdot}) \cong \pi_n(Xj, x_0)$ and $\pi_n^j(X,Y,\varphi_{y_\cdot}) \cong \pi_n(Xj,Yj,y_0)$ which preserve the (evident) group structure when $n \geq 1$ (for the absolute case; the relative case requires $n \geq 2$).

1.3.5 Fact ([Piacenza 2] 2.3 p.816). Let (X,Y) be a pair in J-CGV, $j \in ob\ J$, and $y_0 \in Yj$. Then there exist natural boundary maps ∂ and long exact sequences

$$\cdots \longrightarrow \pi_n^j(X,Y,\varphi_{y_\cdot}) \xrightarrow{\partial} \pi_{n-1}^j(Y,\varphi_{y_\cdot}) \longrightarrow \pi_{n-1}^j(X,\varphi_{y_\cdot}) \longrightarrow \cdots$$
$$\longrightarrow \pi_0^j(Y,\varphi_{y_\cdot}) \longrightarrow \pi_0^j(X,\varphi_{y_\cdot})$$

of groups up to $\pi_1^j(Y,\varphi_{y_\cdot})$ and pointed sets thereafter.

1.3.6 A morphism $f:(X,Y) \longrightarrow (X',Y')$ in J-**CGV** is called a J-**n-equivalence** if for each $j \in ob\ J$, $f(j):(Xj,Yj) \longrightarrow (X'j,Y'j)$ is an n-equivalence in **CGV** (**[Piacenza 2]** p. 816). A morphism f is called a **weak equivalence** if for every $n \geq 0$ it is a J-n-equivalence.

1.3.7 A morphism $f:X \longrightarrow Y$ in J-**CGV** is called a **weak fibration** if for each $j \in ob\ J$, $f(j):Xj \longrightarrow Yj$ is a Serre fibration in **CGV** (**[Piacenza 2]** p. 820).

1.3.8 Fact (**[Piacenza 2]** 5.1 p. 820). A morphism $f:X \longrightarrow Y$ in J-**CGV** is a weak fibration if and only if f has the homotopy lifting property for all objects of the form $D_j \underline{x} I^n$.

1.3.9 A morphism $f:X \longrightarrow Y$ in J-**CGV** is called a **weak cofibration** if f has the left lifting property for any weak fibration that is also a weak equivalence (**[Quillen]**, **[Piacenza 2]**).

1.4 *J-CW* Complexes

1.4.1 Let B^n be the (topological) unit n-ball and S^n the unit n-sphere. Recall that to build a *CW* complex the cells B^n are used. Analogously, to construct an equivariant *CW* complex, G-*CW* complex, cells are of the form $G/H \times B^n$, where G is a group and H is its subgroup. The general approach is to view both cases as an attaching of n-cells of the form $D_r \ \underline{x}\ B^n$ with $r \in$ ob J. Note that according to 1.2.4 and 1.2.6(3) the inclusion

$$\coprod_{\alpha \in \Lambda} D_{j_\alpha} \underline{x}\ S^{n-1} \xrightarrow{\ u\ } \coprod_{\alpha \in \Lambda} D_{j_\alpha} \underline{x}\ B^n$$

is a closed J-cofibration. Let A be a diagram. We attach cells to A by means of the usual push out square:

$$\begin{array}{ccc}
\coprod_{\alpha\in\Lambda} D_{j\alpha} \underline{\times} S^{n-1} & \xrightarrow{u} & \coprod_{\alpha\in\Lambda} D_{j\alpha} \underline{\times} B^{n} \\
\phi \downarrow & & \downarrow \bar{\phi} \\
A & \xrightarrow{\bar{u}} & X .
\end{array}$$

Note if we evaluate the above square for $j \in$ ob J, the resulting square of CGV spaces is a pushout with each u_j a closed cofibration. The relevant facts concerning this are:

(1) Each $\bar{u}_j$ is a closed cofibration and an embedding (1.2.3 & 1.2.5).

(2) $\bar{u}$ is a closed J-cofibration (1.2.8(1)).

1.4.2 A **J-complex** is an object X in J-CGV such that $X = \varinjlim_{p>-1} X^p$, and each X^p is constructed inductively by the pushout diagram

$$\begin{array}{ccc}
\coprod_{\alpha\in\Lambda_p} D_{j\alpha} \underline{\times} S^{p\alpha-1} & \longrightarrow & \coprod_{\alpha\in\Lambda_p} D_{j\alpha} \underline{\times} B^{p\alpha} \\
\downarrow & & \downarrow \\
X^{p-1} & \longrightarrow & X^p
\end{array}$$

with $X^{-1} = \varnothing$. If $X^{-1} = A$ for some object Y in J-CGV, then a pair arising (X,A) is called a **relative J-complex.** A **J-subcomplex** Y of a J- complex X is a J-complex which is a subdiagram of X and whose skeleton Y^p is obtained by attaching some of the cells $D_{j\alpha} \underline{\times} B^{p\alpha}$, $p \in \Lambda_p$. An object X in J-CGV is a ***J-CW* complex** if X is a J-complex and for all $\alpha \in \Lambda_n$, we have $n_\alpha = n$. **Relative J-subcomplexes** and **(relative) *J-CW* subcomplexes** are defined in the usual way. See **[Spanier]** for the standard terminology on *CW*-complexes.

1.4.3 Fact ([Piacenza 2] 3.2 p. 816). Let $j \in$ ob J. Suppose that $e:Y \longrightarrow Z$ is an J-n-equivalence. Then we can complete the following diagram in J-CGV:

$$
\begin{tikzcd}
D_j \times \partial I^n \arrow[rr, "i_0"] \arrow[dd] & & D_j \times \partial I^n \times I \arrow[dl, "h"'] \arrow[dd] & & D_j \times \partial I^n \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"'] & \\
D_j \times I^n \arrow[rr, "i_0"'] \arrow[ur] & & D_j \times I^n \times I \arrow[ul, dotted, "\tilde{h}"] & & D_j \times I^n \arrow[ll, "i_1"] \arrow[ul, dotted, "\tilde{g}"]
\end{tikzcd}
$$

1.4.4 Fact ([Piacenza 2] 3.3 p. 817) (J-HELP). If (X,A) is a relative J-*CW* complex of dimension $\leq$ n and $e:Y \longrightarrow Z$ is a J-n-equivalence then we can complete the following diagram in J-CGV:

$$
\begin{tikzcd}
A \arrow[rr] \arrow[dd] & & A \times I \arrow[dl] \arrow[dd] & & A \arrow[ll] \arrow[dl] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[rr] \arrow[ur] & & X \times I \arrow[ul, dotted] & & X \arrow[ll] \arrow[ul, dotted]
\end{tikzcd}
$$

1.4.5 Fact ([Piacenza 2] 3.4 p. 817) (WHITEHEAD).

(i) Suppose X is a J-*CW* complex, and that $e:Y \longrightarrow Z$ is a J-n-equivalence. Then $e_*:h(X,Y) \longrightarrow h(X,Z)$ is an isomorphism if dim X < n and an epimorphism if dim X = n.

(ii) If $e:Y \longrightarrow Z$ is a weak equivalence, and if X is any J-*CW* complex, then $e_*:h(X,Y) \longrightarrow h(X,Z)$ is an isomorphism.

1.4.6 Fact ([Piacenza 2] 3.5 p. 817) (CELLULAR APPROXIMATION). Suppose that X is a *J-CW* complex, and that A is a sub-*J-CW* complex of X. Then, if f:X $\longrightarrow$ Y is a morphism of *J*-**CGV** which is *J*-cellular when restricted to A, we can homotope f, rel. f|A, to a *J*-cellular morphism g:X $\longrightarrow$ Y.

Denote by $\mathrm{colim}_J(\_)$ or $(\_)/J$ a functor from *J*-**CGV** to **CGV** defined by taking the direct limits of functors. Recall that if X is a topological space, then a subspace $A \subset X$ is called **locally closed** if $A = U \cap F$, where U is open and F is closed. Examples include closed or open subspaces.

1.4.7 Lemma. A locally closed subspace of a **CGV**-space is a **CGV**-space.

Proof. $A=U \cap F$ is a closed subspace of an open subspace U, and then the claim follows from 1.1.2 (3) & (4).

1.4.8 Let X be a diagram over *J*. For each $j \in ob\ J$ let $t_j:X_j \longrightarrow \mathrm{colim}_J X$ be natural maps of X_j into the colimit. Let $\mathbf{A}$ be a subobject of $\mathrm{colim}_J X$ (usually $\mathbf{A}$ is a subspace of $\mathrm{colim}_J X$ taken with k-topology). Let $\check{\mathbf{A}}$ be a pullback of the diagram of **CGV**-spaces:

$$\begin{array}{ccc} & & X_j \\ & & \big\downarrow{\scriptstyle t_j} \\ k(\mathbf{A}) & \longrightarrow & X/J\ , \end{array}$$

or, according to 1.1.2(12), $\check{\mathbf{A}}$ is a *k*-ification of the pullback diagram of **Top**-spaces:

$$\begin{array}{ccc} & & X_j \\ & & \big\downarrow{\scriptstyle t_j} \\ \mathbf{A} & \longrightarrow & X/J\ , \end{array}$$

or we can explicitly put $\check{\mathbf{A}}(j) = k(t_j^{-1}(\mathbf{A}))$, $j \in ob\ J$, and for $i \xrightarrow{s} j$ in *J*

$\check{\mathbf{A}}(s) = Xs|\check{\mathbf{A}}(j)$. Then $\check{\mathbf{A}}$ is a diagram over J, and there is a natural morphism $\check{\mathbf{A}} \longrightarrow X$ (**[Piacenza 2]** p. 817).

1.4.9 Definition. An ordered pair (X, $\mathbf{A}$), where X $\in$ ob J-**CGV** and $\mathbf{A} \subset \operatorname{colim}_J X$, is called a **special pair** if $\operatorname{colim}_J \check{\mathbf{A}} = \mathbf{A}$ and (X, $\check{\mathbf{A}}$) is a J-pair with the embedding arising naturally from the pullback maps (compare **[Piacenza 2]** 3.6 p. 817). Such $\check{\mathbf{A}}$ is called a **special subdiagram** of X. In most situations $\mathbf{A}$ is taken to be open, closed, or locally closed (see 1.4.8). This is due to the following lemma.

1.4.10 Lemma. If $\mathbf{A}$ is a locally closed subspace of $\operatorname{colim}_J X$, then (X, $\mathbf{A}$) is a special pair.

Proof. Since $\mathbf{A}$ is locally closed, $k(\mathbf{A}) = \mathbf{A}$ (1.4.8) and therefore $\check{\mathbf{A}}j = t_j^{-1}(\mathbf{A})$. The colimit of $\check{\mathbf{A}}$, as a diagram of **Top**-spaces, is $\mathbf{A}$, and the assertion follows from commutativity of colim_J and the functor k (compare **[Vogt 1]** Theorem 2.1 and its proof).

1.4.11 Definition. A special pair (X,$\mathbf{A}$) is called **a** J-**neighborhood retract** pair (abbreviated J-NR) if there exist an open subset $\mathbf{U}$ of X/J such that $\mathbf{A} \subseteq \mathbf{U}$ and a retraction morphism $\mathbf{r}:\check{\mathbf{U}} \longrightarrow \check{\mathbf{A}}$ (called a J– **retraction**). A special pair (X,$\mathbf{A}$) is called **a** J-**neighborhood deformation retract** pair (abbreviated J-**NDR**) if (X,$\mathbf{A}$) is a J-NR with a J-retraction $\mathbf{r}$ of $\check{\mathbf{U}}$ to $\check{\mathbf{A}}$ such that if $\mathbf{i}:\check{\mathbf{A}} \subset \check{\mathbf{U}}$, then $\mathrm{id}_{\check{\mathbf{U}}} \simeq \mathbf{i}\circ\mathbf{r}$. A special pair (X, $\mathbf{A}$) is a **local** J–**NDR** if additionally for an arbitrary neighborhood $\mathbf{V}$ of $\mathbf{A}$ an open set $\mathbf{U}$ can be chosen inside $\mathbf{V}$.

From the construction of the operator $(\_)^{\vee}$ and properties of colim_J, the following facts are valid.

1.4.12 Facts ([Piacenza 2] p. 818).

(1) Let X be a *J-CW* complex; that is,

$$X = \varinjlim_{p>-1} X^p, \qquad X^p = \Big(\coprod_{\alpha\in\Lambda} D_{j\alpha} \underline{x}\, B^n\Big)_{\phi} \sqcup X^{p-1}$$

for $p \geq 0$, and $X^{-1} = \varnothing$ or A. The functor colim_J sends cells $D_j \underline{x}\, B^p$ to cells B^p and pushouts X^p in *J*-**CGV** to pushouts $X^p/J = \Big(\coprod_{\alpha\in A} B^n\Big)_{\phi/J} \sqcup (X^{p-1}/J)$ in **CGV** ($p \geq 0$), with $X^{-1}/J = \varnothing$ or A/J. Moreover, $X/J = \varinjlim_{p>-1} (X^p/J)$, so colim_J induces a *CW* structure on X/J. Now X/J has the natural structure of a *CW* complex in **CGV** with all attaching maps being images under colim_J of the corresponding attaching morphisms in *J*-**CGV**. Note that there is a one-to-one correspondence between cells in X and cells in X/J.

(2) Let X be a *J-CW* complex and let A be a subcomplex of X/J. Then $\check{A}$ is a *J*-subcomplex of X.

(3) If A^p is the p-skeleton of X/J, then $(A^p)^{\vee}$ is the p-skeleton of X.

From 1.4.12 the following corollary is immediate.

1.4.13 Corollary. All *J-CW* pairs are special.

1.4.14 Fact (LOCAL CONTRACTIBILITY). Let (X,A) be a special pair in *J*-**CGV** with X a *J-CW* complex and $A = \{a\}$, $a \in X/J$. Then there exits a unique object $j \in J$ such that $\check{A} \cong D_j$ and (X,A) is a local *J*-NDR pair.

Proof. See **[Piacenza 2]** 3.7 p. 818. A proof different from that of Piacenza using collaring techniques is given in 1.6.12.

1.4.15 Fact. Let (X,A) be a *J-CW* pair. Then (X,A) is a *J*-NDR pair.

Proof. See Corollary 1.4.13 and **[Piacenza 2]** 3.8 p. 818.

1.5 Closed Model Category

1.5.1 A **closed model category** is a category M with three distinguished classes of morphisms, which were originally called *cofibrations*, *fibrations*, and *weak equivalences*, and which satisfy appropriate axioms; see **[Quillen]** or **[Baues]** for formulations of the axioms. Piacenza in **[Piacenza 2]** pp. 820-821 showed that a Quillen closed model structure can be defined on *J*-**CGV**; Quillen's cofibrations are defined as weak cofibrations 3.9, fibrations are defined as weak fibrations 3.7 , and weak equivalences are defined as in 3.6. A **trivial weak fibration** is a weak fibration that is also a weak equivalence.

1.5.2 Facts.

(1) (QUILLEN'S FACTORIZATION LEMMA). Any morphism $f:X \longrightarrow Y$ of *J*-**CGV** may be factored $f = p \circ g$ where g is a weak cofibration and p is a trivial weak fibration (**[Piacenza 2]** 5.3 p. 821).

(2) *J*-**CGV** is a closed model category (**[Piacenza 2]** 5.4 p. 821).

(3) Let $X = \varinjlim X_n$, where the limit is taken over a system $\{X_n\}$ of *J*-cofibrations such that each X_n has the *J*-homotopy type of a *J-CW* complex. Then X has the *J*-homotopy type of a *J-CW* complex (**[Piacenza 2]** 5.5 p. 821).

(4) Each J-complex is of the J-homotopy type of a J-CW complex (**[Piacenza 2]** 5.6 p. 822).

(5) (APPROXIMATION THEOREM). There is a functor Γ:J-CGV $\longrightarrow$ J-CGV and a natural transformation $p:\Gamma \longrightarrow \mathrm{id}$ such that for each X $\in$ J-CGV, ΓX is a J-complex, and p_X is a trivial weak fibration (**[Piacenza 2]** 5.7 p. 822).

(6) The category HoJ-CGV is equivalent to the category of J-CW complexes modulo homotopy (**[Piacenza 2]** 5.8 p. 822).

1.6 Collars for J–CW Complexes

This section is devoted to the proof of the Local Contractibility Theorem 1.4.14. All constructions presented in 1.6.1-9 are exhibited and hold for **Top**. Recall that if a diagram of spaces is already in **CGV**, then its colimit in **Top** is a k–space (1.1.2(12)). As a result of the commutativity of the colimit over J with the adjunction construction (pushouts), we will see in 1.6.2 that some inverse image sets (diagrams) have a very simple product structure with one factor being all of D_j and the second being a subset of the unit ball. In fact, the specific manipulations are done inside the unit ball. The relative version of the Lemma on Collaring is obtained by modifying a collar itself instead of changing any characteristic maps.

1.6.1 Let (X, A) be a special J–pair. Assume that (X, $\check{A}$) is the adjunction of n–cells, and let $\bar{f} = \amalg\bar{\phi}_\alpha : \coprod_\alpha D_{j_{\underline{\alpha}}} \times B^n \longrightarrow X$ be the characteristic map and $f = \amalg\phi_\alpha : \coprod_\alpha D_{j_{\underline{\alpha}}} \times S^{n-1} \longrightarrow \check{A}$ be the corresponding attaching map. Then after taking the colimit over J we obtain that a pair $(X/J, A)$ is an adjunction pair of n–cells with the characteristic map $\bar{f}/J = \amalg\bar{\phi}_\alpha/J: \coprod_\alpha B^n_\alpha \longrightarrow X/J$, $B^n_\alpha \cong B^n$ for all α, and with the corresponding attaching map $f/J = \amalg\phi_\alpha/J: \coprod_\alpha S^{n-1}_\alpha \longrightarrow A$, $S^{n-1}_\alpha \cong S^{n-1}$.

1.6.2 Let U be an open subset of A. Then $f/J^{-1}(U) = \coprod_\alpha \phi_\alpha/J^{-1}(U)$ is an open subset of $\coprod_\alpha S^{n-1}_\alpha$, and $f^{-1}(\check{U}) = \coprod_\alpha \phi_\alpha^{-1}(\check{U})$ is a well defined open subdiagram of $\coprod_\alpha D_{j_\alpha} \underline{x} S^{n-1}$. Because the colimit over J commutes with the taking of pushouts, $(\phi_\alpha)^{-1}(\check{U}) = D_{j_\alpha} \underline{x} (\phi_\alpha/J)^{-1}(U)$. Therefore $f^{-1}(\check{U}) = \coprod_\alpha D_{j_\alpha} \underline{x} (\phi_\alpha/J)^{-1}(U)$. Analogous statements hold for a closed subset F of A.

1.6.3 Note that each n–ball B^n can be regarded as a subset of $\mathbb{R}^n$. Therefore a product ty, $t \in [0,1]$, $y \in \coprod_\alpha B^n_\alpha$, is well defined. Similarly, a multiplication of a element (x,y) of $\coprod_\alpha D_{j_\alpha} \underline{x} B^n$ by $t \in [0,1]$, given by $t(x,y) = (x, ty)$, is well defined. Denote $f/J^{-1}(U)\bullet(\frac{1}{2},1] = \{ts\colon s \in f/J^{-1}(U),\ t \in (\frac{1}{2}, 1]\}$ and $f^{-1}(\check{U})\bullet(\frac{1}{2},1] = \{ts\colon s \in f^{-1}(\check{U}),\ t \in (\frac{1}{2},1]\} = \coprod_\alpha D_{j_\alpha} \underline{x} (\phi_\alpha/J)^{-1}(U)\bullet(\frac{1}{2},1]$; note that $f^{-1}(\check{U})\bullet(\frac{1}{2},1]$ is a well defined diagram. Clearly the above construction with $\bullet(\frac{1}{2}, 1]$ depends on a chosen embedding of B^n into $\mathbb{R}^n$.

Denote by $B^n(\frac{1}{2}) = \{x \in B^n \mid \|x\| \le \frac{1}{2}\}$ and $U(\frac{1}{2}) = \{x \in B^n \mid \|x\| > \frac{1}{2}\}$. The radial "retraction" K of B^n to S^{n-1} is defined as the homotopy $K\colon B^n \times I \longrightarrow B^n$ by $K(x,t) = \begin{cases} (1+t)x & \text{if } \|x\| \le 1/(1+t) \\ x/\|x\| & \text{if } \|x\| \ge 1/(1+t) \end{cases}$. Clearly $K_0 = 1_{B^n}$, $K_1(B^n\text{-}\{0\}) \subseteq$

S^{n-1}, and K is stationary on S^{n-1}. The homotopy K restricts on $U(\frac{1}{2})\times I$ to the deformation retraction R of $U(\frac{1}{2})$ to S^{n-1}. Clearly, if $\{B^n_\alpha\}$ is a collection of n–balls, then K induces the homotopy on the disjoint union $\coprod_\alpha B^n_\alpha$.

1.6.4 Lemma on Collaring. Let a pair (X, A) of spaces be an adjunction of n–balls $\{B^n_\alpha\}$ with the attaching maps $\{\psi_\alpha : S^{n-1}_\alpha \longrightarrow A\}$, $t = \amalg\psi_\alpha$. Denote by $\bar{t}$ the characteristic map of the adjunction. Recall that the **$\bar{t}$–collar** is the set $C_{\bar{t}}(U) = U \cup \bar{t}(t^{-1}(U)*(\frac{1}{2},1]) \subseteq X$ (see **[Fritsch - Piccinini]** p. 20). Let U be an open subset of A and let F be a closed subset of A. Then

(1) $C_{\bar{t}}(U) \cap A = U$,

(2) $\bar{t}^{-1}(C_{\bar{t}}(U)) = t^{-1}(U)*(\frac{1}{2}, 1]$,

(3) $C_{\bar{t}}(U)$ is an open subset of X,

(4) There exists a deformation retraction F of $C_{\bar{t}}(U)$ to U induced by the radial deformation retraction R of $t^{-1}(U)*(\frac{1}{2},1]$ to $t^{-1}(U)$ and by the stationary homotopy T of U to U,

(5) $\overline{C_{\bar{t}}(F)} = F \cup \bar{t}(t^{-1}(F)*[\frac{1}{2},1])$,

(6) $C_{\bar{t}}(U)$ is a pushout of U and $t^{-1}(U)*(\frac{1}{2}, 1]$,

(7) A homotopy $G:X\underline{x}I \longrightarrow X$ induced by the homotopy K on $\coprod\{B^n_\alpha\}$ and by the stationary homotopy T on A has the following properties:

(i) $G_0 = 1_X$,

(ii) G is stationary on A,

(iii) $G(C_{\bar{t}}(A)\underline{x}I) \subseteq C_{\bar{t}}(A)$, $G_1(C_{\bar{t}}(A)) \subseteq A$,

(iv) If Y can be obtained by the adjunction of F and some subcollection of $\{B^n_\alpha\}$ (with corresponding characteristic maps ψ_α), then $G(Y\underline{x}I) \subseteq Y$.

Proof. See **[Fritsch-Piccinini]** Lemmas 1.1.7-8 for statements (1)-(6) and

[Brown] Remark 7.3.6 for details on inducing homotopies on adjunction spaces. (7)(i)-(iii) follows again from [Brown], 7.3.6. (7)(i)–(iii) follow directly from the construction of G. For 7(iv) note that $\mathbf{Y}$ is well defined subspace of $\mathbf{X}$ (see [Fritsch-Piccinini], Lemma 1.1.8) and K is mapping any n–ball into itself, and hence K restricts well on the disjoint union of n–balls used to produce $\mathbf{Y}$ (compare [Switzer], Lemma 7.4).

1.6.5 Lemma. If ($\mathbf{X}$; $\mathbf{A}$, $\mathbf{B}$) is a *CW*–triad (of spaces) then there exist an open set $\mathbf{V}$, $\mathbf{A} \subseteq \mathbf{V}$, and a homotopy $G: \mathbf{X} \times I \longrightarrow \mathbf{X}$ satisfying

(i) $G_0 = 1_{\mathbf{X}}$,

(ii) G is stationary on $\mathbf{A}$,

(iii) $G(\mathbf{V}\underline{x}I) \subseteq \mathbf{V}$, $G_1(\mathbf{V}) \subseteq \mathbf{A}$,

(iv) $G(\mathbf{B}\underline{x}I) \subseteq \mathbf{B}$.

Proof. See [Switzer], Lemma 7.4.

1.6.6 Remark. Before presenting the generalization of 1.6.4 and 1.6.5 to the case of diagrams, we discuss a construction of induced subdiagram of a given diagram, and categorical constructions of intersections and unions of two diagrams in the cases when it is possible.

(1) Let X be a diagram and let $\{\underset{a}{A}\}_{a \in \mathrm{ob}J}$ be a family of *k*–spaces such that $\underset{a}{A} \subseteq X_a$ and $X(f)(\underset{b}{A}) \subseteq \underset{a}{A}$, for $f \in J(a,b)$. Then clearly $\tilde{X}_{ab}(\underset{b}{A}\underline{x}J(a,b)) \subseteq \underset{a}{A}$ and thus a continuous map $\tilde{A}_{ab} = \tilde{X}_{ab} | \underset{b}{A}\underline{x}J(a,b) : \underset{b}{A}\underline{x}J(a,b) \longrightarrow \underset{a}{A}$ is is well defined. Therefore a diagram $A: J \longrightarrow$ CGV with $A_a = \underset{a}{A}$ is well defined and is a subdiagram of X.

(2) Let (X; A, B) be a 2–ad of diagrams. Assume that either A or B are

closed or open. Note that the pullback Z of $A \subseteq X \supseteq B$ can be chosen such that each Z_a is $k(A_a \cap B_a)$, $a \in$ ob J (1.1.2(12) and [MacLane 2], p. 111). Since either A or B are closed or open, each $A_a \cap B_a$ is a k-space in the subspace topology (see 1.1.2.(3)-(4)). Therefore Z denoted $A \cap B$ is a well defined subdiagram of A, B, and X with $(A \cap B)_a = A_a \cap B_a$.

(3) Let (X; A, B) be a 2-ad of diagrams. Assume that both A and B are either closed or open. Then $A \cap B$ is a diagram . Note that the pushout W of $A \supseteq A \cap B \subseteq B$ can be chosen such that each W_a is a k-space $A_a \cup B_a$, $a \in$ ob J (1.1.2(12) and the dual statement to [MacLane 2], p. 111). Therefore W denoted $A \cup B$ is a well defined subdiagram of X, and A and B are subdiagrams of $A \cup B$ with $(A \cup B)_a = A_a \cup B_a$.

1.6.7 Definition. Define the $\bar{f}$-collar $C_{\bar{f}}(\check{U})$ of $\check{U}$ to be a subdiagram of X with $C_{\bar{f}}(\check{U})(a) = \check{U}_a \cup \bar{f}_a(f_a^{-1}(\check{U}_a)\bullet(\frac{1}{2}, 1])$, $a \in$ ob J.

1.6.8 Lemma (One-step Collaring for Diagrams). Let a special pair $(X, \mathbb{A})$ be an adjunction of n-cells $\{D_{j_\alpha} \underline{x} B^n\}$. Let $\mathbb{U}$ be an open subset of $\mathbb{A}$ and let $\mathbb{F}$ be a closed subset of $\mathbb{A}$. Then

(1) $C_{\bar{f}}(\check{U}) \cap \check{A} = \check{U}$.

(2) $\bar{f}^{-1}(C_{\bar{f}}(\check{U})) = f^{-1}(\check{U})\bullet(\frac{1}{2},1]$.

(3) $C_{\bar{f}}(\check{U})$ is an open subdiagram of X.

(4) $C_{\bar{f}}(\check{U})$ is a pushout of $\check{U}$ and $f^{-1}(\check{U})\bullet(\frac{1}{2}, 1]$.

(5) $C_{\bar{f}}(\check{U}) = (C_{\bar{f}/J}(\mathbb{U}))\check{}$.

(6) There is a deformation retraction DF_t of $C_{\bar{f}}(\check{U})$ to $\check{U}$ induced by the radial deformation retraction DR of $f^{-1}(\check{U})\bullet(\frac{1}{2}, 1]$ to $f^{-1}(\check{U})$ and the

stationary homotopy DT of $\check{\mathbf{U}}$. The colim$_J$ maps homotopies DR, DT, and DF to R, T, and F, respectively.

(7) $\overline{C_{\bar{f}}(\check{\mathbf{F}})} = \mathbf{F} \cup \bar{f}(f^{-1}(\check{\mathbf{F}}) \bullet [\frac{1}{2}, 1])$.

(8) A homotopy $G: X \underline{x} I \longrightarrow X$ induced by the homotopy K on $\bigsqcup D_{j_\alpha} \underline{x} B^n$ and by the stationary homotopy T on $\check{\mathbf{A}}$ has the following properties:

(i) $G_0 = 1_X$,

(ii) G is stationary on $\check{\mathbf{A}}$,

(iii) $G(C_{\bar{f}}(\check{\mathbf{A}}) \underline{x} I) \subseteq C_{\bar{f}}(\check{\mathbf{A}})$, $G_1(C_{\bar{f}}(\check{\mathbf{A}})) \subseteq \check{\mathbf{A}}$,

(iv) If Y can be obtained by the adjunction of $\check{\mathbf{F}}$ and some subcolection of $\{D_{j_\alpha} \underline{x} B^n\}$ (with corresponding characteristic maps ϕ_α), then $G(Y \underline{x} I) \subseteq Y$.

Proof. (1), (2), and (3) follows directly by checking for each a $\in J$, since each set $C_{\bar{f}}(\check{\mathbf{U}})(a)$ is a pushout of $\check{\mathbf{U}}_a$ and $f_a^{-1}(\check{\mathbf{U}}_a) \bullet (\frac{1}{2}, 1]$ and it is open in X_a. (4), (5) and (7) follow directly by checking for each a $\in$ ob J. (6) and (8) are induced by property of the adjunction.

In obtaining the relative version of the Lemma on Collars for Diagrams, our general approach is to modify the radial neighborhood of $f^{-1}(\check{\mathbf{U}})$ and to keep the characteristic morphisms of adjunctions unchanged.

1.6.9 Lemma (One–step Relative Collaring for Diagrams). Let (X, $\mathbf{A}$) and $\mathbf{U}$ be defined as in Lemma 1.6.8 and moreover assume that there exists $\mathbf{V}$ open in X/J such that $\bar{\mathbf{U}} \subseteq \mathbf{V}$. Then, there exist an open subdiagram $N_{\bar{f}/J}(\mathbf{U}, \mathbf{V})$ of $\bigsqcup_\alpha B^n_\alpha$, an open subdiagram $N_{\bar{f}}(\check{\mathbf{U}}, \check{\mathbf{V}})$ of $\bigsqcup_\alpha D_{j_\alpha} \underline{x} B^n$, and a modified $\bar{f}$–collar $C_{\bar{f}}(\check{\mathbf{U}}, \check{\mathbf{V}})$ such that $C_{\bar{f}}(\check{\mathbf{U}}, \check{\mathbf{V}})(a) = \check{\mathbf{U}}_a \cup \bar{f}_a(N_{\bar{f}}(\check{\mathbf{U}}, \check{\mathbf{V}})(a))$, a $\in$ ob J, and

(0) $\overline{C_{\bar{f}}(\check{U}, \check{V})} \subseteq \check{V}$.

(1) $C_{\bar{f}}(\check{U}, \check{V}) \cap \check{A} = \check{U}$.

(2) $\bar{f}^{-1}(C_{\bar{f}}(\check{U}, \check{V})) = N_{\bar{f}}(\check{U}, \check{V})$.

(3) $C_{\bar{f}}(\check{U}, \check{V})$ is an open subdiagram of X.

(4) $C_{\bar{f}}(\check{U}, \check{V})$ is a pushout of $\check{U}$ and $N_{\bar{f}}(\check{U}, \check{V})$.

(5) $C_{\bar{f}}(\check{U}, \check{V}) = (C_{\bar{f}/J}(U, V))^{\vee}$, where $C_{\bar{f}/J}(U, V)$ is an adjunction of U and an open subdiagram $N_{\bar{f}/J}(U, V)$ of $\coprod_{\alpha} B^n_{\alpha}$. The diagram $N_{\bar{f}/J}(U, V)$ is a subdiagram of $f/J^{-1}(U)*(\frac{1}{2}, 1]$ and admits a radial retraction R to $f/J^{-1}(U)$ induced from that of $f/J^{-1}(U)*(\frac{1}{2},1]$ to $f/J^{-1}(U)$.

(6) There is a deformation retraction DF of $C_{\bar{f}}(\check{U}, \check{V})$ to $\check{U}$ induced by the radial deformation retraction DR of $N_{\bar{f}}(\check{U}, \check{V})$ to $\bar{f}^{-1}(\check{U})$ and the stationary homotopy DT of $\check{U}$. colim_J maps DR, DT, and DF to R, T, and F, respectively.

(7) A homotopy $G: X \times \underline{I} \longrightarrow X$ induced by the homotopy K on $\coprod D_{J_{\alpha}} \underline{x} B^n$ and by the stationary homotopy T on $\check{A}$ has the following properties:

(i) $G_0 = 1_X$,

(ii) G is stationary on $\check{A}$,

(iii) $G(C_{\bar{f}}(\check{U}, \check{V})\underline{x}\underline{I}) \subseteq C_{\bar{f}}(\check{U}, \check{V})$, $G_1(C_{\bar{f}}(\check{U}, \check{V})) \subseteq \check{A}$,

(iv) If Y can be obtained by the adjunction of $\check{F}$ and some subcolection

Proof. First note that $\bar{f}^{-1}(\check{V}) = \coprod_{\alpha} D_{J_{\alpha}}\underline{x}(\bar{\phi}_{\alpha}/J)^{-1}(V)$ and $\bar{f}^{-1}(\check{U}) = \coprod_{\alpha}$.

Proof. First note that $\bar{f}^{-1}(\check{V}) = \coprod_{\alpha} D_{J_{\alpha}}\underline{x}(\bar{\phi}_{\alpha}/J)^{-1}(V)$ and $\bar{f}^{-1}(\bar{U}) = \coprod_{\alpha} D_{J_{\alpha}}\underline{x}(\phi_{\alpha}/J)^{-1}(\bar{U})$. Since $S^{n-1}_{\alpha} \supseteq (\phi_{\alpha}/J)^{-1}(\bar{U}) \subseteq (\bar{\phi}_{\alpha}/J)^{-1}(V) \subseteq B^n_{\alpha}$ with $(\bar{\phi}_{\alpha}/J)^{-1}(V)$ open in B^n_{α}, there is an open subset M_{α} of B^n_{α} which can be radially deformed to $M_{\alpha} \cap S^{n-1}_{\alpha}$ and such that $\overline{M_{\alpha}} \subseteq (\bar{\phi}_{\alpha}/J)^{-1}(V)$, $(\phi/J)^{-1}(\bar{U}) \subseteq M_{\alpha} \cap S^{n-1}_{\alpha}$. Define $N_{\bar{\phi}_{\alpha}/J}(U, V) = \{x: x/|x| \in (\phi_{\alpha}/J)^{-1}(U)\}$. Note that

$N_{\bar{\phi}_\alpha/J}(\mho, \mathcal{V})$ is open in $B^n_{\bar{\alpha}}$, $\overline{N_{\bar{\phi}_\alpha/J}(\mho, \mathcal{V})} \subseteq (\bar{\phi}_\alpha/J)^{-1}(\mathcal{V})$, and it can be radially deformed to $(\phi_\alpha/J)^{-1}(\mho)$. Moreover we may assume that $N_{\bar{\phi}_\alpha/J}(\mho, \mathcal{V}) \subseteq (\phi_\alpha/J)^{-1}(\mho)\bullet(\frac{1}{2},1]$ with the retraction homotopy induced from $(\phi_\alpha/J)^{-1}(\mho)\bullet(\frac{1}{2},1]$. Therefore replacing $(\phi_\alpha/J)^{-1}(\mho)\bullet(\frac{1}{2},1]$ by $N_{\bar{\phi}_\alpha/J}(\mho, \mathcal{V})$ in adjunctions of Lemma 1.6.6 yields the claim. Clearly, $N_{\bar{f}/J}(\mho, \mathcal{V}) = \coprod_\alpha N_{\bar{\phi}_\alpha/J}(\mho, \mathcal{V})$ and $N_{\bar{f}}(\check{\mho}, \check{\mathcal{V}}) = \coprod_\alpha D_{J_\alpha} x N_{\bar{\phi}_\alpha/J}(\mho, \mathcal{V})$ have the required properties.

So far, we obtained our results by redirecting manipulations to the unit ball B^n, all done for a case of finite *J-CW* complexes. In following part we shall prove similar result for infinite *J-CW* complexes.

1.6.10 Definitions ([Fritsch - Piccinini] Appendix).

(1) Suppose a set V is open (closed) in X iff each $V \cap U_\lambda$ is open (closed) in U_λ, for ever λ; this is equivalent to require that a function $f{:}X \longrightarrow Y$ is continuous iff each $f|U_\lambda$ is continuous for every λ (p. 246). Then the topology of a space X is said to be **determined** by a family of subspaces $\{U_\lambda\}$.

(2) Suppose a function $f{:}X \longrightarrow Z$, where Z is any space, is continuous iff the compositions $f \circ f_\lambda{:}Y_\lambda \longrightarrow Z$ are continuous for all λ (p. 246). Then the space X is said to have the **final topology** with respect to the family of maps $\{f_\lambda{:}Y_\lambda \longrightarrow X\}$.

(3) An **expanding sequence** of spaces is a sequence $\{X_n\colon n \in \mathbb{N}\}$ of spaces such that, for every $n \in \mathbb{N}$, X_n is a subspace of X_{n+1}, and every inclusion,

$X_n \subseteq X_{n+1}$, is a closed cofibration (p. 273).

(4) The **union space** of the expanding sequence is the space $X = \bigcup_{n=0}^{\infty} X_n$ endowed with the final topology with respect to the family of inclusions $X_n \subseteq X$. Then, all of the X_n are closed subspaces of X and X is determined by the family $\{X_n: n \in \mathbb{N}\}$ (p. 273).

1.6.11 Useful theorems from [Fritsch - Piccinini], Appendix.

(1) **Proposition A.5.1 (i)&(iii)** (p. 273). Let $\{X_n: n \in \mathbb{N}\}$ be an expanding sequence and let X be its union space. Then

(a) For every space Z, the sequence $\{X_n \times Z: n \in \mathbb{N}\}$ is an expanding sequence with union space $X \times Z$ (**Remark: this claim requires the usage of the exponential law from CGV**).

(b) The inclusions $X_n \subseteq X$ are closed cofibrations, for every $n \in \mathbb{N}$.

(2) Let $\{X_n: n \in \mathbb{N}\}$ be an expanding sequence with union space X. Let Z be a space and $\{g_n: n \in \mathbb{N}\}$ be a sequence of maps $g_n: X \longrightarrow Z$ such that $g_{n+1} \simeq g_n$ rel. X_n. Then, the map $g: Z \longrightarrow Z$ defined by $g|X_n = g_n|X_n$, for every $n \in \mathbb{N}$, is homotopic to g_0 rel. X_0 (**Proposition A.5.7**, p. 277).

(3) Let $\{X_n: n \in \mathbb{N}\}$ be an expanding sequence with union space X and such that X_n is a strong deformation retract of X_{n+1}, for every $n \in \mathbb{N}$. Then, X_0 is a strong deformation retract of X (**Corollary A.5.8**, p. 277).

1.6.12 Definition of Infinite Collar (a reformulation from **[Fritsch - Piccinini]**, p. 27). Let X be a (possible infinite) *J-CW* complex, and let $\{f^n\}$ be a sequence of characteristic maps for the adjunctions (X^n, X^{n-1}). Assume that $\mathbf{A} = X^m/J$, for some $m \in \mathbb{N}$. Let $\mathbf{U}$ be an open subset of $\mathbf{A}$ and $\mathbf{V}$ an open subset of X/J such that $\bar{\mathbf{U}} \subseteq \mathbf{V}$. An infinite collar $C^{\infty}(\check{U}, \check{V})$ is defined as follows. For every $n \geq m$, define $\check{U}^{n+1} = C_{\bar{f}^{n+1}}(\check{U}^{n}, \check{V})$, and then take $C^{\infty}(\check{U}, \check{V}) = \operatorname{colim}_{n \geq m} \check{U}^{n}$.

1.6.13 Proposition. Let all X, **A**, **U**, and **V** be defined as above. Then an infinite collar $C^{\infty}(\check{U}, \check{V})$ satisfies:

(1) $C^{\infty}(\check{U}, \check{V})$ intersects X^{m} in $\check{U}$,

(2) $C^{\infty}(\check{U}, \check{V})$ is an open subdiagram of X,

(3) each $C^{\infty}(\check{U}, \check{V})(a)$, $a \in \operatorname{ob} J$, is the union space of an expanding sequence of spaces,

(4) contains $\check{U}$ as the strong deformation retract.

Moreover

(5) A homotopy $G: X\underline{x}I \longrightarrow X$ induced by the homotopy K on each $\coprod D_{j_\alpha}\underline{x}B^{n}$, $n \geq m$, and by the stationary homotopy T on $\check{A}$ has the following properties:

(i) $G_0 = 1_X$,

(ii) G is stationary on $\check{A}$,

(iii) $G(C_{\bar{f}}(\check{U}, \check{V})\underline{x}I) \subseteq C_{\bar{f}}(\check{U}, \check{V})$, $G_1(C_{\bar{f}}(\check{U}, \check{V})) \subseteq \check{A}$,

(iv) If Y can be obtained by the adjunction of $\check{F}$, where **F** is closed subset of **A**, and subcolections of $\{D_{j_\alpha}\underline{x}B^{n}\}_{\alpha \in \Lambda_n}$, $n \geq m$, with the corresponding characteristic maps ϕ_α, then $G(Y\underline{x}I) \subseteq Y$.

Proof. The proof is in essence done by for checking the above properties for each $a \in \operatorname{ob} J$. (1) is obvious from the construction since

$$C^{\infty}(\check{U}, \check{V}) \cap X^{n} = \begin{cases} C_{\bar{f}^{n}}(\check{U}^{n-1}, \check{V}) & \text{if } n > m \\ \check{U} & \text{if } n = m \end{cases}.$$

(2) Let $a \in \operatorname{ob} J$. For $n > m$, the intersections $C^{\infty}(\check{U}, \check{V})_a \cap X^{n}_a = C_{\bar{f}^{n}}(\check{U}^{n-1}, \check{V})(a)$ are open in X^{n}_a and note that X_a has the colimit topology

with respect to $\{X^n_*\}$.

(3) Each inclusion $C_{\bar f^n}(\check U^{n-1}, \check V) \subseteq C_{\bar f^{n+1}}(\check U^n, \check V) = C_{\bar f^{n+1}}(C_{\bar f^n}(\check U^{n-1}, \check V), \check V)$ is a closed cofibration.

We show that each $C^\infty(\check U, \check V)_*$ has the colimit topology with respect to $\{C_{\bar f^n}(\check U^{n-1}, \check V)\}$. Let $W \subseteq C^\infty(\check U, \check V)$ be a diagram such that $W \cap C_{\bar f^n}(\check U^{n-1}, \check V)$ is open in $C_{\bar f^n}(\check U^{n-1}, \check V)$. Since $C_{\bar f^n}(\check U^{n-1}, \check V)$ is open in X^n, $W \cap C_{\bar f^n}(\check U^{n-1}, \check V)$ is open in X^n. Note that $W \cap C_{\bar f^n}(\check U^{n-1}, \check V) = W \cap X^n$, and thus W is open in X, so it is open in $C^\infty(\check U, \check V)$.

(4) The claim follows from 1.6.11(3) (**[Fritsch - Piccinini]** Appendix, Corollary A.5.8).

1.6.14 Proposition (LOCAL CONTRACTIBILITY THEOREM). Let $(X, \mathbf{A})$ be a special pair in *J*–CGV with X a *J–CW* complex and $\mathbf{A} = \{\mathbf{a}\}$, $\mathbf{a} \in X/J$. Then there exists a unique object $j \in$ ob J such that $\check{\mathbf{A}} \cong D_j$ and $(X, \mathbf{A})$ is a local *J*–NDR pair.

Proof. Let V be an open neighborhood of $\mathbf{a}$ in X/J. We show that there is an open subdiagram Y of X which is contractible to $\check{\mathbf{A}}$. Since X/J is a *CW* complex, the point $\mathbf{a}$ is in the interior of some m–ball B^n_j, which in turn corresponds to a representable diagram D_j, for some $j \in$ ob J. Clearly, $\check{\mathbf{A}} = D_j \underline{\times} \{a\} \cong D_j$. Since $\mathbf{a} \in \mathrm{int}(B^n_j)$, there exists an open neighborhood U of $\mathbf{a}$ in $\mathrm{int}(B^n_j)$, which is contractible to $\mathbf{a}$, and such that $\bar U \subseteq V$. Clearly $\check U = D_j \underline{\times} U$ is *J*–contractible to $\check{\mathbf{A}}$. An application of 1.6.13 gives us the existence of an open, infinite collar $C^\infty(\check U, \check V)$ which is still contained in $\check V$. Take $Y = C^\infty(\check U, \check V)$. The contractibility of $C^\infty(\check U, \check V)$ to $\check U$ followed by the contractibility of $\check U$ to $\check{\mathbf{A}}$ gives us the assertion.

Note that if $\mathbf{U}$ is inside a cell of the induced *CW*–structure on X/J, then $\check{\mathbf{U}} = D_{\underline{j}} x \mathbf{U}$. In such case $\{x_0\}^{\vee} = D_{\underline{j}} x \{x_0\}$ is a strong deformation retract of $\check{\mathbf{U}} = D_{\underline{j}} x \mathbf{U}$.

1.6.15 Lemma. If $(X; A, B)$ is a *J–CW*–triad then there exist an open subdiagram V of X, $A \subseteq V$, and a homotopy $G: X \times I \longrightarrow X$ satisfying

(i) $G_0 = 1_X$,

(ii) G is stationary on A,

(iii) $G(V\underline{x}I) \subseteq V$, $G_1(V) \subseteq A$,

(iv) $G(B\underline{x}I) \subseteq B$.

Proof. Compare with [Switzer], Lemma 7.4. Let $\mathbf{A} = A/J$ and $\mathbf{B} = B/J$. Both $\mathbf{A}$ and $\mathbf{B}$ are closed subsets of X/J and thus $\mathbf{F} = \mathbf{A} \cap \mathbf{B}$ is a closed subset. Since X can be obtained from A by the successive attaching of cells, we apply 1.6.13.

CHAPTER 2

AXIOMATIC HOMOLOGY AND COHOMOLOGY FOR DIAGRAMS

We shall assume a knowledge of the axiomatic approach as can be found in the book of Eilenberg and Steenrod, "Foundation of Algebraic Topology". Section 2.1 contains the definition of an admissible category which will serve as a domain for homology and cohomology theories. Pairs of diagrams in an admissible category are subject to a set of axioms modeled on the Eilenberg-Steenrod approach. The condition on one–point spaces is replaced by the analogous condition on representable functors. Sections 2.2 and 2.3 contain a formulation of axioms for homology and cohomology for diagrams. These axioms are a reformulation of the classical Eilenberg–Steenrod axioms for homology and cohomology. The dimension axiom is formulated for representable functors.

Section 2.4 contains the proof of the uniqueness of homology on the category of finite *J–CW* complexes. With the additional additivity assumption on homology, we obtain the uniqueness theorem for *J–CW* arbitrary complexes. The uniqueness theorems show that homology theory, as in the classical case, when restricted to (finite) *J–CW* complexes, is completely determined by the its values for representable functors. Section 2.5 contains the analogous results for cohomology theories.

2.1. Admissible Categories

This section contains the definition of the admissible category of pairs of diagrams following the Eilenberg-Steenrod approach.

2.1.1 Definition. An **admissible category** $\mathcal{Ad}$ of pairs of diagrams is defined as a subcategory of the category of pairs of J-CGV satisfying the following six conditions:

(1) If (X,A) is in $\mathcal{Ad}$, then the pairs (∅, ∅), (A, ∅), (X, ∅), (A, A), (X, A), and (X, X) together with all induced inclusion morphisms between them, lie in $\mathcal{Ad}$. These pairs and inclusions are called the **lattice** of the pair (X, A).

(2) If f:(X,A) $\longrightarrow$ (Y,B) is in $\mathcal{Ad}$, then $\mathcal{Ad}$ contains all morphisms induced by f between lattices of (X, A) and (Y, B).

(3) If (X,A) is in $\mathcal{Ad}$ and I = [0,1] is the unit closed interval, then the product pair (X, A)$\underline{x}$I = (X$\underline{x}$I, A$\underline{x}$I) is in $\mathcal{Ad}$, together with morphisms g_0, g_1:(X, A) $\longrightarrow$ (X$\underline{x}$I, A$\underline{x}$I) given by $g_0(x) = (x,0)$ and $g_1(x) = (x,1)$.

We say that a diagram X is in $\mathcal{Ad}$ if the pair (X,∅) is in $\mathcal{Ad}$. Similarly, a morphism of diagrams f:X $\longrightarrow$ Y is in $\mathcal{Ad}$ if the induced morphism of pairs f:(X,∅)$\longrightarrow$(Y,∅) is in $\mathcal{Ad}$.

(4) All representable diagrams, D_j, are in $\mathcal{Ad}$.

(5) If for some $j \in$ ob J, a morphism f:Dj $\longrightarrow$ X is in J-CGV and X is in $\mathcal{Ad}$, then f is in $\mathcal{Ad}$.

(6) If f:X $\longrightarrow$ Y is a map of finite CW-complexes, $j \xrightarrow{h} i$ in J, then $D_i\underline{x}X \xrightarrow{\hom(\_,h)xf} D_j\underline{x}Y$ is in $\mathcal{Ad}$.

2.1.2 J-pairs and their morphisms in $\mathcal{A}d$ are called **admissible.** We say that a triple (resp. an n-tuple or n-ad) of diagrams (and their morphisms) is admissible if all resulting J-pairs are admissible. Moreover, two admissible morphisms are homotopic in $\mathcal{A}d$ if there is a homotopy between them which is admissible.

2.1.3 Examples of Admissible Categories

(1) J-**CGV**^{2}. It is closed on tensoring and cotensoring by spaces and morphisms from **CGV** (see 1.1.10 & 1.1.11).

(2) $\mathcal{A}d_S$ is the category of all special pairs and their morphisms.

(3) $\mathcal{A}d_{FCW}$ is the category of pairs of finite *J-CW* complexes and their morphisms.

(4) $\mathcal{A}d_{CW}$ is the category of pairs of *J-CW* complexes and their morphisms.

(5) $\mathcal{A}d_{LC}$ is the category of all J-pairs (X,A) and their morphisms such that colim_J gives rise to the pairs ($\mathrm{colim}_J X$, $\mathrm{colim}_J A$) in **LocCHaus.**

(6) A J-pair (X,A) is **saturated** if $X^{-1}(f)(A_a) = A_b$ for all a,b $\in$ ob J, f $\in$ J(a,b). Denote by $\mathcal{A}d_{Sat}$ the category of saturated J-pairs and their morphisms.

2.1.4 Let (X,A) be a J-pair. Let a $\in$ ob J, and let $f{:}j \longrightarrow i$ in J. Then each X_a-A_a is a subspace of X_a. If it is always true that $X(f)(X_i\text{-}A_i) \subseteq X_j\text{-}A_j$, then we can define the complement of A in X as a (difference) diagram $X\text{-}A{:}J^{op} \longrightarrow$ **Top** with $(X\text{-}A)(a) = X_a\text{-}A_a$ and $(X\text{-}A)(f) = X(f)|X_i\text{-}A_i$. The difference diagram X-A as a diagram taking values in **Top** can be naturally regarded as a subdiagram of X. Note that that happens in the case of saturated diagrams (see 2.1.3(6)). Clearly, if each X_a-A_a, with the subspace topology, is already a k-space, then X-A is a J-diagram and moreover (X,X-A) is a J-pair. Similarly, if (X;A,B) is a 2-ad in J, then the assignments $(A \cap B)(a) = A_a \cap B_a$ with the subspace topology, and $(A \cap B)(f) = X(f) | A_i \cap B_i$ defines a (intersection) diagram taking values in **Top**. If each $A_a \cap B_a$, with the subspace topology, is already a k-space, then the intersection $A \cap B$ is a J-diagram and moreover $(X, A \cap B)$, $(A, A \cap B)$, and $(B, A \cap B)$ are J-pairs.

2.1.5 Lemma (On Saturated and Special Pairs).

(1) Let U be an open (resp. closed) saturated subdiagram of X and let A be a subdiagram of X. Then the difference A-U is a well defined subdiagram of A and X. Similarly, the intersection $A \cap U$ is a well defined subdiagram of A, U, and X. Since for a special pair (X, $\mathfrak{U}$), the induced J-pair (X, $\check{\mathfrak{U}}$) is a saturated pair, the claim holds for special pairs too.

(2) If U and A are saturated subdiagrams of X with U open, then A–U is a saturated subdiagram of X.

(3) Let (X, $\mathfrak{U}$) be special pair with $\mathfrak{U}$ locally closed. Then, for any subdiagram A of X, the intersection $A \cap \check{\mathfrak{U}}$ is a well defined subdiagram of A, $\check{\mathfrak{U}}$, and X.

Proof. For (1) we note that for $a \in$ ob J, sets $U_a \cap A_a$ and A_a-U_a with the subspace topology from A_a, are already k-spaces as closed or open subsets of A_a. The part (3) follows from observation that each $\check{U}_a \cap A_a$ is a locally closed subset of A_a and therefore a k-space (compare 1.4.8).

2.1.6 Definition. Let A and B be two subdiagrams of X. We say that A is contained in the interior of B if, for every $a \in \text{ob } J$, $A_a \subseteq \text{int}_{X_a} B_a$.

2.1.7 Definition. $\mathcal{E}$ is called a category of admissible supports if $\mathcal{E}$ is a full subcategory of $\mathcal{A}d$ which is closed on taking finite unions, that is, if (X_1,A_1), (X_2,A_2),...,(X_m,A_m) are subdiagrams of (X,A) in $\mathcal{A}d$ and each (X_i,A_i) is in $\mathcal{E}$, then $(X_1,A_1)\cup(X_2,A_2)\cup...\cup(X_m,A_m) = (X_1\cup X_2\cup...\cup X_m,\ A_1\cup A_2\cup...\cup A_m)$ is in $\mathcal{E}$.

2.1.8 Definition. A pair $(\mathbf{C}, \mathbf{D})$ in **CGV** is called a **compact pair** if $\mathbf{C}$ is Hausdorff compact and $\mathbf{D}$ is closed in $\mathbf{C}$. Let $(X, \mathbf{A})$ be a special pair. A compact pair $(\mathbf{C}, \mathbf{D}) \subset (X/J, \mathbf{A})$ is called a **special compact (sub)pair** of $(X, \mathbf{A})$. From 1.4.10 follows that $(X, \mathbf{C})$ is a special pair, and $(\check{\mathbf{C}}, \check{\mathbf{D}})$ is a J-subpair of $(X, \check{\mathbf{A}})$.

2.2 Axiomatic Homology

2.2.1 Let $\mathcal{A}d$ be an admissible category of pairs of diagrams. A **homology theory** on $\mathcal{A}d$ consists of a covariant functor H_* from $\mathcal{A}d$ to a graded abelian category (usually with **Ab** or ${}_R$**Mod**) and a natural transformation ∂_* from H_* on pairs (X, A) to H_* restricted to pairs $(X,\emptyset)$, satisfying the following axioms. We set $H_*(X)$ for $H_*(X,\emptyset)$.

2.2.2 HOMOTOPY AXIOM. If $f,g:(X,A) \longrightarrow (Y,B)$ are homotopic in $\mathcal{A}d$, then $H_*f = H_*g$.

2.2.3 EXACTNESS AXIOM. For any pair (X,A) in $\mathcal{Ad}$ with inclusion morphisms $i:A\longrightarrow X$ and $j:(X,\varnothing)\longrightarrow(X,A)$ there is an exact sequence

$$\ldots\longrightarrow H_{q+1}(X,A)\xrightarrow{\partial_{q+1}(X,A)}H_q(A)\xrightarrow{H_q i}H_q(X)\xrightarrow{H_q j}H_q(X,A)\longrightarrow\ldots\ .$$

2.2.4 EXCISION AXIOM. Let (X,A) be a pair in $\mathcal{Ad}$. Let U be an open saturated subdiagram of X whose closure $\bar{U}$ is contained in the interior of A. Then the inclusion morphism $u:(X\text{-}U,A\text{-}U)\longrightarrow(X,A)$, if admissible, induces an isomorphism $H_*u:H_*(X\text{-}U,A\text{-}U)\longrightarrow H_*(X,A)$.

2.2.5 EXCISION AXIOM FOR SPECIAL PAIRS. Let $(X,\mathbb{A})$ be a special pair in $\mathcal{Ad}$. Let $\mathbb{U}$ be an open subspace of X/J whose closure $\bar{\mathbb{U}}$ is contained in the interior of $\mathbb{A}$. Then the inclusion morphism $u:(X\text{-}\check{\mathbb{U}},\check{\mathbb{A}}\text{-}\check{\mathbb{U}})\longrightarrow(X,\check{\mathbb{A}})$, if admissible, induces an isomorphism $H_*u:H_*(X\text{-}\check{\mathbb{U}},\check{\mathbb{A}}\text{-}\check{\mathbb{U}})\longrightarrow H_*(X,\check{\mathbb{A}})$.

2.2.6 DIMENSION AXIOM. For every representable functor D_j, $H_q(D_j)=0$ for $q\neq 0$.

We have the following Supplementary Axioms.

2.2.7 AXIOM OF $\mathcal{C}$–SUPPORTS. Given any pair (X,A) in $\mathcal{Ad}$ and given $z\in H_q(X,A)$ there is a pair (X',A') in $\mathcal{C}$ such that (X',A') is a subpair of (X,A) and z is in the image of H_*u, where u is a inclusion morphism $(X',A')\subset(X,A)$.

2.2.8 AXIOM OF COMPACT SUPPORTS FOR SPECIAL PAIRS. Given any special pair $(X,\mathbb{A})$ in $\mathcal{Ad}$ and given $z\in H_q(X,\check{\mathbb{A}})$ there is a special compact subpair

$(\check{X}',\check{A}') \subset (X/J,A)$ in $\mathcal{Ad}$ such that z is in the image of H_*u, where u is a inclusion morphism $(\check{X}',\check{A}') \subset (X,\check{A})$.

2.2.9 ADDITIVITY AXIOM. If X is a coproduct of $\{X_\alpha\}$ in $\mathcal{Ad}$ with inclusion morphisms $u_\alpha : X_\alpha \longrightarrow X$, then the morphism $\sum_\alpha u_{\alpha *} : \sum_\alpha H_*(X_\alpha) \longrightarrow H_*(X)$ is an isomorphism.

2.3 Axiomatic Cohomology

2.3.1 Let $\mathcal{Ad}$ be an admissible category of pairs of diagrams. A **cohomology theory** on $\mathcal{Ad}$ consists of a contravariant functor H^* from $\mathcal{Ad}$ to a graded abelian category (usually with **Ab** or ${}_R$**Mod**) and a natural transformation ∂^* from H^* on pairs (X, A) to H^* restricted to pairs $(X,\varnothing)$, satisfying the following axioms. We set $H^*(X)$ for $H^*(X,\varnothing)$.

2.3.2 HOMOTOPY AXIOM. If $f,g:(X,A) \longrightarrow (Y,B)$ are homotopic in $\mathcal{Ad}$, then $H^*f = H^*g$.

2.3.3 EXACTNESS AXIOM. For any pair (X,A) in $\mathcal{Ad}$ with inclusion morphisms $i:A \longrightarrow X$ and $j:(X,\varnothing) \longrightarrow (X,A)$ there is an exact sequence

$$\ldots \longrightarrow H^q(X,A) \xrightarrow{H^q j} H^q(X) \xrightarrow{H^q i} H^q(A) \xrightarrow{\partial^q(X,A)} H^{q+1}(X,A) \longrightarrow \ldots .$$

2.3.4 EXCISION AXIOM. Let (X,A) be a pair in $\mathcal{Ad}$. Let U be an open saturated subdiagram of X whose closure $\bar{U}$ is contained in the interior of A. Then the inclusion morphism u:(X-U,A-U) $\longrightarrow$ (X,A), if admissible, induces an isomorphism H^*u:H^*(X,A) $\longrightarrow$ H^*(X-U,A-U).

2.3.5 EXCISION AXIOM FOR SPECIAL PAIRS. Let $(X,\check{A})$ be a special pair in $\mathcal{Ad}$. Let $\mathbf{U}$ be an open subspace of X/*J* whose closure $\bar{\mathbf{U}}$ is contained in the interior of $\mathbf{A}$. Then the inclusion morphism $u:(X-\check{\mathbf{U}},\check{A}-\check{\mathbf{U}}) \longrightarrow (X,\check{A})$, if admissible, induces an isomorphism $H^*u:H^*(X,\check{A}) \longrightarrow H^*(X-\check{\mathbf{U}},\check{A}-\check{\mathbf{U}})$.

2.3.6 DIMENSION AXIOM. For every representable functor D_j, $H^q(D_j) = 0$ for $q \neq 0$.

We add the following Supplementary Axiom.

2.3.7 ADDITIVITY (WEDGE) AXIOM. If X is a coproduct of $\{X_\alpha\}$ in $\mathcal{Ad}$ with inclusion morphisms $u_\alpha:X_\alpha \longrightarrow X$, then the morphism $\prod_\alpha u_\alpha^*:H^*(X) \longrightarrow \prod_\alpha H^*(X_\alpha)$ is an isomorphism.

2.4 Uniqueness of Homology

The purpose of the present section is to show that on the category of finite *J-CW* complexes homology theories are determined, up to isomorphism, by their coefficient systems.

2.4.1 Homology of Finite Coproducts

Let (X,**A**) be a special pair. Assume that $X = X_1 \amalg X_2$. Then each X_i is a special subdiagram of X, and an induced separation of A is given by $\mathbf{A}_i = \mathbf{A} \cap X_i/J$ (i = 1, 2). The following proposition holds for any homology theory satisfying the Exactness and Excision axioms.

Proposition. Let (X, **A**) be an admissible special pair with X being the coproduct $X = X_1 \amalg X_2$. If $u_1:(X_1,\check{\mathbf{A}}_1) \subseteq (X, \check{\mathbf{A}})$ and $u_2:(X_2,\check{\mathbf{A}}_2) \subseteq (X,\check{\mathbf{A}})$ are the inclusions induced by the coproduct, then

$u_{1*}\oplus u_{2*}:H_*(X_1, \check{\mathbf{A}}_1) \oplus H_*(X_2, \check{\mathbf{A}}_2) \longrightarrow H_*(X, \check{\mathbf{A}})$ is an isomorphism.

Proof. Note that $X_1\cup\check{\mathbf{A}}$ is well defined and $X_1\cup\check{\mathbf{A}} = X_1 \amalg \check{\mathbf{A}}_2 = (X_1/J \cup \mathbf{A}_2)^{\vee}$, and $(X, X_1\cup\check{\mathbf{A}}, \check{\mathbf{A}})$ is a special triple. For any homology theory satisfying the Exactness Axiom there is the homology sequence of the triple (see **[Spanier]** 4.8.5 p.201):

$$\ldots\rightarrow H_q(X_1\cup \check{\mathbf{A}}, \check{\mathbf{A}}) \xrightarrow{i_*} H_q(X, \check{\mathbf{A}}) \xrightarrow{j_*} H_q(X, X_1\cup \check{\mathbf{A}}) \xrightarrow{\partial_*} H_{q-1}(X_1\cup \check{\mathbf{A}}, \check{\mathbf{A}}) \rightarrow\ldots$$

The inclusion $k:(X_2, \check{\mathbf{A}}_2) \longrightarrow (X, X_1\cup\check{\mathbf{A}})$ is an excision of X_1/J which is open and closed in X/J, and similarly, $t:(X_1, \check{\mathbf{A}}_1) \longrightarrow (X_1\cup\check{\mathbf{A}}, \check{\mathbf{A}})$ is an excision of $\check{\mathbf{A}}_2/J$ which is open and closed in $X_1/J \amalg \mathbf{A}_2$. Therefore both k_* and t_* are isomorphisms. Since $k = j \circ i_2$ and $k_* = j_* \circ i_{2*}$, $i_{2*}\circ(k_*)^{-1}:H_*(X, X_1\cup\check{\mathbf{A}}) \longrightarrow H_*(X, \check{\mathbf{A}})$ is a splitting of the long exact sequence.

Let {H, ∂} be a homology theory defined on some admissible category $\mathcal{A}d$. For each $j \in$ ob J we define a homology theory $\{{}^jH, {}^j\partial\}$ on a subcategory of CGV^2 consisting of pairs (X, A) such that $D_j\underline{x}(X, A) \in \mathcal{A}d$ by putting

$$ {}^{j}H_*(X,A) = H_*\left(D_j \underline{\times}\, (X, A)\right), \qquad {}^{j}H_*(f) = H_*(\mathrm{id}_{D_j} \underline{\times}\, f), \text{ and} $$

$$ {}^{j}\partial_*(X,A) = \partial_*\left(D_j \underline{\times}\, (X, A)\right). $$

Note that $\{{}^{j}H, {}^{j}\partial\}$ is defined on the category of finite *CW* complexes and their maps.

Let $G_j = H_0(D_j)$ for $j \in$ ob J. In fact, G is a (covariant) coefficient system on J, that is, a covariant functor from J to **Ab** (or ${}_R$**Mod**). Define $Gj = H_*(D_j)$ for $j \in$ ob J, and $Gf = H_*(\hom(\_,f))$ for $f \in$ morph J. Then G is called the **coefficient system** of H_*.

2.4.2 Proposition. For each object j of J, $\{{}^{j}H_*, {}^{j}\partial_*\}$ is a classical homology theory with coefficient group $G_j = H_0(D_j)$.

Proof. Direct calculations show that $\{{}^{j}H, {}^{j}\partial\}$ satisfies the corresponding Eilenberg-Steenrod axioms with the coefficient group G_j.

In a view of 2.4.2, all classical theorems hold for $\{{}^{j}H_*, {}^{j}\partial_*\}$, with the coefficient group $H_0(D_j)$, and in particular we have the following corollary.

2.4.3 Corollary. Let $j \in$ ob J. For $n \geq 0$, we have

$$ H_q\left(D_j \underline{\times}\, (B^n, S^{n-1})\right) \cong \begin{cases} H_0(D_j) & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases} $$

Proof. Apply [Hu], I.8.1, to ${}^{j}H_q(B^n, S^{n-1}) = H_q\left(D_j \underline{\times}\, (B^n, S^{n-1})\right)$.

Let (X,A) be a *J-CW* pair. Assume that X is obtained from a diagram A by the adjunction of $m > 0$ n-dimensional cells. Let $p_i.D_{j_i} \underline{\times}\, (B^n, S^{n-1}) \longrightarrow (X,A)$ $(i = 1,2,\ldots,m)$ be the characteristic morphisms of those cells.

2.4.4 Proposition. The induced homomorphisms

$$p_{i*}: H_*\left(D_{j_i}\underline{x}\ (B^n,\ S^{n-1})\right) \longrightarrow H_*(X,A) \quad (i = 1,2,\dots,m) \text{ are monomorphisms}$$

and

$$\bar\phi_* = \oplus p_{i*}: \bigoplus_{i=1}^{m} H_*\left(D_{j_i}\underline{x}\ (B^n, S^{n-1})\right) \longrightarrow H_*(X,A) \quad \text{is an isomorphism.}$$

Proof. Denote $\bar\phi = \coprod p_i$. The claim follows from the diagram

$$\begin{tikzcd}
\coprod_i D_{j_i}\underline{x}\ (B^n,\ S^{n-1}) \arrow[r, "\bar\phi"] \arrow[d, "(1)"] & (X,\ A) \arrow[d, "(2)"] \\
\coprod_i D_{j_i}\underline{x}\ (B^n,\ B^n-\{0\}) \arrow[r, "\bar\phi"] & (X,\ X-\bar\phi(\coprod_i D_{j_i}\underline{x}\{0\})) \\
\coprod_i D_{j_i}\underline{x}\ (B^n(\tfrac{1}{2}),\ B^n(\tfrac{1}{2})-\{0\}) \arrow[r, "\bar\phi"] \arrow[u, "(3)"] & (X-C_{\bar f}(A),\ X-C_{\bar f}(A)-\bar\phi(\coprod_i D_{j_i}\underline{x}\{0\})), \arrow[u, "(4)"]
\end{tikzcd}$$

where $B^n(\frac{1}{2}) = \{x \in B^n:\ \|x\| \le \frac{1}{2}\}$. The inclusions (1) and (2) induce isomorphisms in homology because $D_j\underline{x}B^n$ is a strong deformation retract of $D_j\underline{x}(B^n-\{0\}) = \left(B^n_j-\{0\}\right)^\vee$, and A is a strong deformation retract of $X-\bar\phi(\coprod_i D_{j_i}\underline{x}\{0\}) = \left(X/J - \bar\phi/J(\bigcup_i\{0_i\})\right)^\vee$. The inclusions (3) and (4) are excisions of open sets U_i, each $U_i = \{x \in B^n:\ \|x\| > \frac{1}{2}\}$ and resp. $V = C_{\bar f/J}(A/J)$ (see 1.6.4, Lemma on Collars). Note that $\left(C_{\bar f/J}(A/J)\right)^\vee = C_{\bar f}(A)$ and all pairs in the above diagram are special. Note that the bottom morphism is not only an inclusion but it is an isomorphism.

2.4.5 Proposition. If (X,A) is a finite *J-CW* pair of relative dimension n, then

$$H_q(X,A) = 0$$

for every $q < 0$ and $q > n$.

Proof. The following proof is a formal modification of Hu's proof (**[Hu]**, Proposition 8.4).

If $\dim(X,A) = -1$, then $A = X$ and $H_*(X,A) = 0$ from the long exact sequence of $(X,A) = (X,X)$.

Assume that the claim holds for every finite *J–CW* pair of relative dimension less than n. Let Y be the relative (n-1)–skeleton of (X,A). Then $\dim(Y,A) < n$ and therefore the induction hypothesis holds for (Y,A). Moreover X is obtained from Y by the adjunction of n–cells and satisfies the claim by 2.4.4. The homology long exact sequence of the triple (X, Y, A) has the form:

$$\ldots \longrightarrow H_{q+1}(X,Y) \xrightarrow{\bar{\partial}} H_q(Y,A) \xrightarrow{i_*} H_q(X,A) \xrightarrow{j_*} H_q(X,Y) \xrightarrow{\bar{\partial}} H_{q-1}(Y,A) \longrightarrow \ldots$$

If $q < 0$, then from the inductive hypothesis $H_q(Y,A) = 0$ and $H_{q-1}(Y,A) = 0$, and therefore $H_q(X,A) \cong H_q(X,Y) = 0$.

If $q > n$, then $H_{q+1}(X,Y) = 0$ and $H_q(X,Y) = 0$, and hence $H_q(X,A) \cong H_q(Y,A) = 0$.

2.4.6 Let $\mathcal{K} = \{H_*, \partial_*\}$ and $\mathcal{K}' = \{H_*', \partial_*'\}$ be two homology theories defined on admissible categories $\mathcal{Ad}$ and $\mathcal{Ad}'$ each containing the category $\mathcal{Ad}_{FC}$ of finite *J-CW* complexes. Denote by $G = H_*(D\_)$ and $G' = H_*'(D\_)$ their coefficient systems.

A homomorphism between two coefficient systems G and G′ is a natural transformation between functors G and G′.

A homomorphism between two homology theories, say from $\mathcal{K}$ to $\mathcal{K}'$ defined on some common admissible category, is a natural transformation h from H_* to H_*' commuting with ∂_* and ∂_*'. That is, for both theories and every admissible pair (X,A), we have $h_{q-1}(X) \circ \partial_q(X,A) = \partial'_q(X,A) \circ h_q(X,A)$. Since representable diagrams D_j are admissible for both homology theories, any homomorphism between them induces a homomorphism between their coefficient systems.

2.4.7 Proposition. Let $h:G \longrightarrow G'$ be an arbitrary homomorphism between coefficient systems of homology theories $\mathcal{K}$ and $\mathcal{K}'$ defined on some admissible category containing $\mathcal{A}d_{FCW}$. Then, there is a unique homomorphism $\tilde{h}$ from $\mathcal{K}|\mathcal{A}d_{FCW}$ to $\mathcal{K}'|\mathcal{A}d_{FCW}$ which induces h.

Proof. The proof is a formal modification of Hu's proof ([Hu], Th. 9.1, p.51). Here we shall present the construction of $\tilde{h}_*:H_*(X,A) \longrightarrow H'_*(X,A)$ by induction on the relative dimension dim(X,A).

Case dim(X,A) = -1. X = A and from the Exactness axiom both $H_*(X,A) = 0$ and $H'_*(X,A) = 0$. Therefore, the zero homomorphism is the only possible extension of h.

Case dim(X,A) = 0. X is obtained from A by attaching finitely many 0–cells, say $D_{j_1}xv_1,\dots, D_{j_m}xv_m$. From 2.4.4 both $\bar{\phi}_* = \oplus p_{i*}:\bigoplus_{i=1}^{m} H_*\left(D_{j_i}xv_i\right) \longrightarrow H_*(X,A)$ and $\bar{\phi}'_* = \oplus p'_{i*}:\bigoplus_{i=1}^{m} H'_*\left(D_{j_i}xv_i\right) \longrightarrow H'_*(X,A)$ are isomorphisms attachings). After identifying $H_*\left(D_{j_i}\right)$ with $H_*\left(D_{j_i}xv_i\right)$ and $H'_*\left(D_{j_i}\right)$ with $H'_*\left(D_{j_i}xv_i\right)$ via canonical isomorphisms, we define (uniquely) $\tilde{h}_0 = \bar{\phi}_0 \circ h \circ (\bar{\phi}_0^{-1})$ and $\tilde{h}_q = 0$ for $q \neq 0$.

Case dim(X,A) > 0. Assume that $\tilde{h}_*$ has been constructed for every finite J–CW pair of relative dimension less than n = dim(X,A). Before pursuing the general case we first consider two special subcases. Let $j \in$ ob J. Note that $(D_{\underline{j}}\underline{x}B^n, D_{\underline{j}}\underline{x}S^{n-1})$ is a *J–CW* pair of relative dimension 1. Choose $v_0 \in S^{n-1}$. Then $(D_{\underline{j}}\underline{x}S^{n-1}, D_{\underline{j}}\underline{x}v_0)$ is a *J–CW* pair of relative dimension n-1. Because $D_{\underline{j}}\underline{x}v_0$ is a *J*–deformation retract of $D_{\underline{j}}\underline{x}B^n$, $H_*(D_{\underline{j}}\underline{x}S^{n-1}, D_{\underline{j}}\underline{x}v_0) = 0$ and $H'_*(D_{\underline{j}}\underline{x}S^{n-1}, D_{\underline{j}}\underline{x}v_0) = 0$. The exact homology sequences of the triple $(D_{\underline{j}}\underline{x}B^n, D_{\underline{j}}\underline{x}S^{n-1}, D_{\underline{j}}\underline{x}v_0)$ taken in both homology theories give us the following commutative ladder, where both $\bar{\partial}$ and $\bar{\partial}'$ are isomorphisms.

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & H_q(D_{\underline{j}}\underline{x}B^n, D_{\underline{j}}\underline{x}S^{n-1}) \arrow[r, "\bar{\partial}"] \arrow[d] & H_{q-1}(D_{\underline{j}}\underline{x}S^n, D_{\underline{j}}\underline{x}v_0) \arrow[r] \arrow[d, "\tilde{h}_{q-1}"] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & H'_q(D_{\underline{j}}\underline{x}B^n, D_{\underline{j}}\underline{x}S^{n-1}) \arrow[r, "\bar{\partial}'"] & H'_{q-1}(D_{\underline{j}}\underline{x}S^n, D_{\underline{j}}\underline{x}v_0) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Therefore $\tilde{h}_q : H_q(D_{\underline{j}}\underline{x}S^{n-1}, D_{\underline{j}}\underline{x}S^{n-1}) \longrightarrow H'_q(D_{\underline{j}}\underline{x}S^{n-1}, D_{\underline{j}}\underline{x}S^{n-1})$ can be uniquely defined as $\tilde{h}_q = (\bar{\partial}'^{-1}) \circ \tilde{h}_{q-1} \circ \bar{\partial}$.

Assume now that (X,A) is an adjunction of n–dimensional cells. Then we have natural isomorphisms:

$$\underset{i}{\oplus}\bar{\phi}_{i*} : \bigoplus_{i=1}^{m} H_*\left[D_{\underline{j}_i}\underline{x}(B^n, S^{n-1})\right] \longrightarrow H_*(X,A) \quad \text{and}$$

$$\underset{i}{\oplus}\bar{\phi}'_{i*} : \bigoplus_{i=1}^{m} H'_*\left[D_{\underline{j}_i}\underline{x}(B^n, S^{n-1})\right] \longrightarrow H'_*(X,A).$$

From the previous part $\tilde{h}_*$ is already defined on each $D_{\underline{j}_i}\underline{x}\,(B^n, S^{n-1})$, and denoted ${}^i\tilde{h}_*$. Define $h_* : H_*(X,A) \longrightarrow H'_*(X,A)$ as the composition $(\underset{i}{\oplus}\bar{\phi}'_{i*}) \circ (\underset{i}{\oplus}{}^i\tilde{h}_*) \circ (\underset{i}{\oplus}\bar{\phi}_{i*})^{-1}$ and note that h_* is unique.

Let (X,A) be an arbitrary *J–CW* complex pair of relative dimension dim(X,A) = n. For the relative (n-1)–skeleton Y of (X,A), we have dimension dim(Y,A) < n an therefore $\tilde{h}_* = \hat{h}$ is already uniquely constructed for (Y,A).

Similarly (X, Y) is an adjunction of n-cells and thus $\bar{h}_* = \bar{h}_*:H_*(X,Y) \longrightarrow H'_*(X,Y)$ is already uniquely defined. The following diagram is induced by long exact sequences of the triple (X, Y, A).

$$
\begin{tikzcd}
\dots \arrow[r] & H_{q+1}(X,Y) \arrow[r, "\bar{\partial}"] \arrow[d, "\hat{h}_{q+1}"] & H_q(Y,A) \arrow[r, "i_*"] \arrow[d, "\bar{h}_q"] & H_q(X,A) \arrow[r, "j_*"] \arrow[d] & H_q(X,Y) \arrow[r] \arrow[d, "\hat{h}_q"] & \dots \\
\dots \arrow[r] & H'_{q+1}(X,Y) \arrow[r, "\bar{\partial}'"] & H'_q(Y,A) \arrow[r, "i'_*"] & H'_q(X,A) \arrow[r, "j'_*"] & H'_q(X,Y) \arrow[r] & \dots
\end{tikzcd}
\tag{1}
$$

If $q \neq n, n\text{-}1$, then 2.4.3 and 2.4.4 yield $H_q(X,Y) = 0$, $H_{q+1}(X,Y) = 0$, $H'_q(X,Y) = 0$, $H'_{q+1}(X,Y) = 0$, and the above ladder around takes the form

$$
\begin{tikzcd}
0 \arrow[r] & H_q(Y,A) \arrow[r, "i_*"] \arrow[d, "\bar{h}_q"] & H_q(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H'_q(Y,A) \arrow[r, "i'_*"] & H'_q(X,A) \arrow[r] & 0,
\end{tikzcd}
$$

where both i_*, i'_* are isomorphisms. Therefore, we define the unique $\bar{h}_q:H_q(X,A) \longrightarrow H'_q(X,A)$ by $\bar{h}_q = i'_*\circ\bar{h}_q\circ(i_*)^{-1}$.

If $q = n\text{-}1$, then $H_{n-1}(X,Y) = 0$ and $H'_{n-1}(X,Y) = 0$, and thus the diagram (1) takes the form:

$$
\begin{tikzcd}
\dots \arrow[r] & H_n(X,Y) \arrow[r, "\bar{\partial}"] \arrow[d, "\hat{h}_{q+1}"] & H_{n-1}(Y,A) \arrow[r, "i_*"] \arrow[d, "\bar{h}_q"] & H_{n-1}(X,A) \arrow[r, "j_*"] \arrow[d] & 0 \\
\dots \arrow[r] & H'_n(X,Y) \arrow[r, "\bar{\partial}'"] & H'_{n-1}(Y,A) \arrow[r, "i'_*"] & H'_{n-1}(X,A) \arrow[r, "j'_*"] & 0 .
\end{tikzcd}
$$

Since the rows are exact and the rectangle on the left commutates, there is the unique map between cokernels of $\bar{\partial}$ and $\bar{\partial}'$: $\tilde{h}_q: H_{n-1}(X,A) \longrightarrow H'_{n-1}(X,A)$.

If $q = n$, then $H_{n+1}(X,Y) = 0$, $H_n(Y,A) = 0$, $H'_{n+1}(X,Y) = 0$, and $H'_n(Y,A) = 0$ and the diagram (1) yields:

$$\begin{array}{ccccccc}
0 & \xrightarrow{i_*} & H_n(X,A) & \xrightarrow{j_*} & H_n(X,Y) & \xrightarrow{\bar{\partial}} & H_{n-1}(Y,A) \\
 & & \downarrow & & \downarrow \hat{h}_{q+1} & & \downarrow \tilde{h}_q \\
0 & \xrightarrow{i'_*} & H'_n(X,A) & \xrightarrow{j'_*} & H'_n(X,Y) & \xrightarrow{\bar{\partial}'} & H'_{n-1}(Y,A)\,.
\end{array}$$

Since the rows are exact and the rectangle on the right commutates, there is the unique map between kernels of j_* and j'_*: $\tilde{h}_q: H_n(X,A) \longrightarrow H'_n(X,A)$. This completes the inductive construction of $\tilde{h}_*$.

Uniqueness of the homomorphism between homology theories (both restricted to $\mathcal{Ad}_{FCW}$) gives us the following corollary.

2.4.8 Corollary (The Uniqueness Theorem). If a homomorphism $\tilde{h}$ between homology theories on some admissible category containing $\mathcal{Ad}_{FCW}$ induces an isomorphism between their coefficient systems, then $\tilde{h}\,|\,\mathcal{Ad}_{FCW}$ is itself an isomorphism.

Assume that the homology theory $\mathcal{K} = \{H_*, \partial_*\}$ satisfies the Axiom of Compact Supports for Special Pairs. Then, as in **[Spanier]** p. 204, we have the following proposition.

2.4.9 Proposition. Let $\mathcal{K}$ be a homology theory with compact supports and let $(\mathbf{X}', \mathbf{A}')$ be a special compact subpair of $(X, \mathbf{A})$. Given $z \in H_q(\check{\mathbf{X}}', \check{\mathbf{A}}')$ in the kernel of $H_q(\check{\mathbf{X}}', \check{\mathbf{A}}') \longrightarrow H_q(X, \check{\mathbf{A}})$, there is a special compact pair $(\mathbf{X}'', \mathbf{A}'')$ of $(X, \mathbf{A})$, with $(\mathbf{X}', \mathbf{A}') \subseteq (\mathbf{X}'', \mathbf{A}'') \subseteq (X/J, \mathbf{A})$, such that z is in the kernel of $H_q(\check{\mathbf{X}}', \check{\mathbf{A}}') \longrightarrow H_q(\check{\mathbf{X}}'', \check{\mathbf{A}}'')$.

Proof. The proof is a formal modification of Spanier's proof (**[Spanier]** Lemma 4.8.12, p. 204) obtained by replacing compact pairs by special compact pairs. This is possible since special compact subpairs of a given special pair are closed upon taking finite unions.

2.4.10 Proposition. A homology theory $\mathcal{K} = \{H_*, \partial_*\}$ has compact supports for special pairs if and only if for any special pair $(X, \mathbf{A})$, the induced homomorphism $i: \varinjlim \{H(\check{\mathbf{X}}', \check{\mathbf{A}}') \mid (\mathbf{X}', \mathbf{A}')$ is a special compact pair of $(X, \mathbf{A})\}$ $\longrightarrow H(X, \check{\mathbf{A}})$ is an isomorphism.

Proof. The proof is a formal modification of Spanier's proof (**[Spanier]** 4.8.13, p. 204).

2.4.11 Proposition. Let h be a homomorphism from K to $\mathcal{K}'$ which is an isomorphism for coefficient systems. If $\mathcal{K}$ and $\mathcal{K}'$ have compact supports for special pairs, h is an isomorphism for any *J-CW* pair.

Proof. First note that colim_J and the operation $(\_)^{\vee}$ gives a one-to-one correspondence between special compact subpairs of a special pair $(X, \mathbf{A})$ and compact pairs in $(X/J, \mathbf{A})$. Furthermore they preserve the inclusion relations. Since finite *CW* subcomplexes of $(X/J, \mathbf{A})$ are cofinal in the family of all compact pairs in $(X/J, \mathbf{A})$, finite *J-CW* subcomplexes of $(X, \check{\mathbf{A}})$ are cofinal in the family of all special compact pairs in $(X,\check{A})$. The theorem follows from 2.4.7 and 2.4.10 as in [Spanier] (Theorem 4.8.14, p. 205).

2.5 Uniqueness of Cohomology

In this section we list results on cohomology parallel to those in Section 2.6. The proofs are dual.

2.5.1 Proposition (Cohomology of Finite Coproducts). Let $(X, \check{A})$ be an admissible special pair with X being the coproduct $X = X_1 \amalg X_2$. If $u_1:(X_1,\check{A}_1) \subseteq (X, \check{A})$ and $u_2:(X_2,\check{A}_2) \subseteq (X,\check{A})$ are the inclusions induced by the coproduct, then $u_1^*\oplus u_2^*: H^*(X, \check{A}) \longrightarrow H^*(X_1, \check{A}_1) \oplus H^*(X_2, \check{A}_2)$ is an isomorphism.

Let $\{H^*, \delta^*\}$ be a cohomology theory defined on some admissible category $\mathcal{Ad}$. For each $j \in$ ob J define a cohomology theory $\{{}^jH^*, {}^j\delta^*\}$ on a subcategory of $\mathbf{CGV}^2$ consisting of pairs (X, A) such that $D_j\underline{\times}(X, A) \in \mathcal{Ad}$ by putting

$$ {}^jH^*(X,A) = H^*\left(D_j\underline{\times}\,(X, A)\right), \qquad {}^jH^*(f) = H^*(\mathrm{id}_{D_j}\underline{\times}\, f), \text{ and} $$

$$ {}^j\delta^*(X,A) = \delta^*\left(D_j\underline{\times}\,(X, A)\right). $$

Note that $\{{}^jH^*, {}^j\partial^*\}$ is defined on the category of finite *CW* complexes and their maps.

Put $G_j = H^0(D_j)$ for $j \in$ ob J. In fact, G is a (contravariant) coefficient system on J, that is, a contravariant functor from J to **Ab** (or ${}_R$Mod). $Gj = H^*(D_j)$ for $j \in$ ob J, and $Gf = H^*(\hom(\_,f))$ for $f \in$ morph J. G is called the **coefficient system** of H^*.

2.5.2 Proposition. For each object j of J, $\{{}^jH^*, {}^j\delta^*\}$ is a classical cohomology theory with the coefficient group $G_j = H^0(D_j)$.

2.5.3 Corollary. Let $j \in$ ob J. For $n \geq 0$, we have

$$H^q\left(D_j \underline{\times} (B^n, S^{n-1})\right) \cong \begin{cases} H^0(D_j) & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is dual to that of [Hu], I.8.1, with ${}^jH^q(B^n, S^{n-1}) = H^q\left(D_j \underline{\times} (B^n, S^{n-1})\right)$.

Let X be obtained from a diagram A by adjunction of m>0 n-dimensional cells. Let $p_i : D_{j_i} \underline{\times} (B^n, S^{n-1}) \longrightarrow (X,A)$ $(i = 1,2,\ldots,m)$ be the characteristic morphisms of those cells.

2.5.4 Proposition. The induced homomorphisms

$$p_i^* : H^*(X,A) \longrightarrow H^*\left(D_{j_i} \underline{\times} (B^n, S^{n-1})\right) \quad (i = 1,2,\ldots,m)$$

are epimorphisms and $\bar{\phi}^* = \times_i p_i^* : H^*(X,A) \longrightarrow \bigoplus_{i=1}^{m} H^*\left(D_{j_i} \underline{\times} (B^n, S^{n-1})\right)$ is an isomorphism.

Proof. The proof is dual to that of 2.4.4.

2.5.5 Proposition. Let $h : G \longrightarrow G'$ be arbitrary homomorphism between coefficient systems of cohomology theories $\mathcal{K}$ and $\mathcal{K}'$ defined on some admissible category containing $\mathcal{Ad}_{FCW}$. Then, there is a unique homomorphism $\tilde{h}$

from $\mathcal{K}|\mathcal{A}d_{FCW}$ to $\mathcal{K}'|\mathcal{A}d_{FCW}$ which induces h.

Proof. The proof is dual to that of 2.4.6.

Uniqueness of the homomorphism between cohomology theories (both restricted to $\mathcal{A}d_{FCW}$) gives us the following corollary.

2.5.6 Corollary (The Uniqueness Theorem). If a homomorphism $\tilde{h}$ between cohomology theories on some admissible category containing $\mathcal{A}d_{FCW}$ induces an isomorphism between their coefficient systems, then $\tilde{h}|\mathcal{A}d_{FCW}$ is itself an isomorphism.

2.6 Mayer-Vietoris Sequence for Diagrams

Recall that a triad of diagrams (X; A, B) is a triple of diagrams such that A and B are subdiagrams of X, $X_a = A_a \cup B_a$ for $a \in$ ob J, and $A \cap B$ is a subdiagram of A and B (see 1.6.6(2)).

2.6.1 Definition. A triad of diagrams (X; A, B) is called **excisive** with respect to homology theory $\mathcal{K} = \{H_*, \partial_*\}$ if the inclusion $e:(A, A\cap B) \subseteq (X, B)$ induces an isomorphism $e_*:H_*(A, A\cap B) \longrightarrow H_*(X, B)$

In next two propositions we shall prove every closed saturated J–triad (X; A, B) (A, B are closed saturated subdiagrams of X with $X_a = A_a \cup B_a$ for $a \in$ ob J) such that $\operatorname{int}(A_a) \cup \operatorname{int}(B_a) = X_a$ for $a \in$ ob J is excisive as well as every J–CW–triad is excisive.

2.6.2 Proposition (compare [Switzer], 7.2 and 7.3). For every closed saturated J–triad $(X; A, B)$ (A, B are closed saturated subdiagrams of X with $X_a = A_a \cup B_a$ for $a \in$ ob J) such that $\operatorname{int}(A_a) \cup \operatorname{int}(B_a) = X_a$ for $a \in$ ob J the inclusion $e:(A, A\cap B) \subseteq (X, B)$ induces an isomorphism:

$$e_*:H_*(A, A\cap B) \longrightarrow H_*(X, B).$$

Proof. Compare [Switzer], 7.3. Note that X-A is an open subdiagram and $\overline{X_a - A_a} \subseteq X_a - \operatorname{int}(A_a) \subseteq \operatorname{int}(B_a)$ for $a \in$ ob J. Since $\big(X-(X-A), B-(X-A)\big) = (A, A\cap B)$, the Excision Axiom yields the claim.

2.6.3 Proposition. For every J–CW–triad $(X; A, B)$ the inclusions $e:(A, A\cap B) \subseteq (X, B)$ induces an isomorphism:

$$e_*:H_*(A, A\cap B) \longrightarrow H_*(X, B).$$

Proof. The proof is a modification of the classical proof for CW complexes as given in [Lundell - Weingram], Theorem V.1.1. For a given J–CW–triad $(X;B,A)$ there exists an open subdiagram V of X, $B \subseteq V$, and a homotopy $G:X\times I \longrightarrow X$ as in Lemma 1.6.13, and thus the inclusion morphisms $i_1:(A, A\cap B) \subseteq (A, A\cap V)$ and $i_2:(X,B) \longrightarrow (X,V)$ induces isomorphisms in homology. We have the commutative diagram:

$$\begin{array}{ccc} H_*(A, A\cap B) & \xrightarrow{j_*} & H_*(X,B) \\ {\scriptstyle i_{1*}}\downarrow \cong & & \cong \downarrow {\scriptstyle i_{2*}} \\ H_*(A, A\cap V) & \xrightarrow[j'_*]{} & H_*(X,V) \end{array}$$

Note that X-A is an open subdiagram and X-$A \subseteq B \subseteq V$ and thus $\overline{X-A} \subseteq \operatorname{int}(V)$.

Since $(X-(X-A),\ V-(X-A)) = (A, A\cap V)$, the Excision Axiom gives us that $j'_*: H_*(A, A\cap V) \longrightarrow H_*(X,V)$. is an isomorphism. Therefore j_* is an isomorphism.

2.6.4 Theorem. If $(X\colon A, B)$ is an excisive triad of diagrams and $C \subseteq A\cap B$, then there is an exact sequence

$$\ldots \xrightarrow{\Delta} H_q(A\cap B, C) \xrightarrow{i_*} H_q(A,C)\oplus H_q(B,C) \xrightarrow{j_*} H_q(X,C) \xrightarrow{\Delta} H_{q-1}(A\cap B, C) \longrightarrow \ldots$$

where $i_*(z) = (i_{1*}(z), i_{2*}(z))$, $j_*(x,y) = i_{3*}(x) - i_{4*}(y)$ and Δ is the composite

$$H_q(X,C) \xrightarrow{J_*} H_q(X,B) \underset{\cong}{\xleftarrow{e_*}} H_q(A, A\cap B) \xrightarrow{\bar{\partial}} H_{q-1}(A\cap B, C).$$

Here i_1, i_2, i_3, i_4, e, J are the inclusions

$$\begin{tikzcd} & (A,\ C) \arrow[dr, "i_3"] & \\ (A\cap B,\ C) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & (X,\ C), \\ & (B,\ C) \arrow[ur, "i_4"'] & \end{tikzcd} \qquad \begin{array}{c} (A,\ A\cap B) \xrightarrow{e} (X,\ B) \\ \\ (X,\ C) \xrightarrow{J} (X,\ B) \end{array}$$

and $\bar{\partial}$ is the boundary operator for the triple $(A, A\cap B, C)$.

Proof. The proof is a formal modification of the proof given in **[Switzer]**, Theorem 7.16.

The following corollaries are applications of the Mayer-Vietories sequence, proved by the formal modification of classical proofs given in [Switzer]..

2.6.5 Corollary. If $(X\colon A, B)$ is an excisive triad of diagrams, then the inclusions $i_A:(A, A\cap B) \longrightarrow (X, A\cap B)$, $i_B:(B, A\cap B) \longrightarrow (X, A\cap B)$ induce an isomorphism

$$i_{A*}\oplus i_{B*}:H_*(A, A\cap B)\oplus H_*(B, A\cap B) \longrightarrow H_*(X, A\cap B).$$

2.6.6 Corollary. If $X = X_1 \amalg X_2$, then the inclusions $i_1:X_1 \longrightarrow X$, $i_2:X_2 \longrightarrow X$ induce an isomorphism

$$i_{1*}\oplus i_{2*}:H_*(X_1)\oplus H_*(X_2) \longrightarrow H_*(X).$$

2.6.7 Corollary. If $(X, A) = (X_1, A_1) \amalg (X_2, A_2)$, then the inclusions $i_1:(X_1, A_1) \longrightarrow (X, A)$, $i_2:(X_2, A_2) \longrightarrow (X, A)$ induce an isomorphism

$$i_{1*}\oplus i_{2*}:H_*(X_1, A_1)\oplus H_*(X_2, A_2) \longrightarrow H_*(X, A).$$

CHAPTER 3

SINGULAR J–HOMOLOGY AND J–COHOMOLOGY

This chapter contains a generalization of Illman's approach from G–spaces, over the orbit category of G, to diagrams of topological spaces over a small topological category. The applied apparatus follows the tone of the preprint of [Vogt 2] but the specific constructions presented in this chapter deviate much from the very abstract approach in [Vogt 2] involving categories enriched in chain complexes. Also, the sketchy character of Vogt's preprint does not give a convincing argument that his singular homology and cohomology for diagrams satisfy all the axioms.

In this chapter we expand and reformulate constructions from [Illman], in the purely algebraic way. We do not apply much more than tensoring and homing of (co)chain complexes, taking (co)kernels and we intensively modify the constructions from classical singular homology theory (see [Spanier]).

Section 3.1 introduces our notation for (co)chain complexes of modules over a commutative unitary ring. In Section 3.2 we define the coefficient systems as continuous functors on J taking values in the category of (co)chain complexes. In sections 3.2 and 3.3 "singular" (co)chain complexes are constructed for any saturated J–pair. The homology of those complexes gives us, by definition, "singular" homology and cohomology of diagrams. In short, the singular functor is applied to the topological category J, to "components" of the diagram and to its adjoint maps. In a process similar to that of taking the coend and end of modules over a differential graded category, the required (co)chain complexes are obtained. The Exactness Axiom is established immediately from the construction. Sections 3.5 to 3.13 are devoted to the rest of the axioms.

The existence of induced morphisms between pairs of diagrams is proved in sections 3.5 and 3.6. The Homotopy Axiom is verified in sections 3.7 and 3.8. Sections 3.9 and 3.10 are devoted to the Excision Axiom. The classical subdivision operator for singular simplexes is generalized to the case of diagrams. All constructions follow the pattern of the classical singular theory, as presented in **[Spanier]**. The method of proving the Dimension Axiom (sections 3.11 and 3.12) gives the very pleasant results on the singular (co)chain complex of D_j, a representable functor. In Section 3.13 we show that the Additivity Axiom is satisfied.

3.1 Generalities on Complexes of Modules

3.1.1 If S and R are rings let ${}_S\mathbf{Compl}_R$ (resp. ${}_S\mathbf{CoCompl}_R$) denote the category of chain complexes (resp. cochain complexes) of (S–R)-modules. R will always denote a commutative unitary ring and $\otimes = \otimes_{\mathbf{Z}}$. If $A \in \mathbf{Mod}_{\mathbf{Z}}$ and $B \in {}_{\mathbf{Z}}\mathbf{Mod}_R$, then $A \otimes B = A \otimes_{\mathbf{Z}} B$ has the structure of a right R-module (**[Rotman]** 1.8). If

$A \in \mathbf{Mod}_{\mathbf{Z}}$ and $B \in {}_{R}\mathbf{Mod}_{\mathbf{Z}}$, then $\mathrm{Hom}(A,B) = \mathrm{Hom}_{\mathbf{Z}}(A,B)$ has the structure of a left R-module (**[Rotman]** 1.15(iv)). Let $\mathbb{A}$ be a chain complex of (right) $\mathbf{Z}$-modules and $\mathbb{B}$ be a chain complex of right R-modules. The chain complex $\mathbb{A}\otimes\mathbb{B} = \mathbb{A}\otimes_{\mathbf{Z}}\mathbb{B}$ is given by $[\mathbb{A}\otimes\mathbb{B}]_n = \sum_{p+q=n} A_p \otimes B_q$ with differential

$$\partial^{\otimes}(a\otimes b) = \partial^{A}a\otimes b + (-1)^{|a|}a\otimes\partial^{B}b.$$

Let $\mathbb{A}$ be a chain complex of (right) $\mathbf{Z}$-modules and $\mathbb{B}$ be a cochain complex of left R-modules. The cochain complex $\underline{\mathrm{Hom}}(\mathbb{A},\mathbb{B}) = \underline{\mathrm{Hom}}_{\mathbf{Z}}(\mathbb{A},\mathbb{B})$ is given by $[\underline{\mathrm{Hom}}(\mathbb{A},\mathbb{B})]_n = \prod_{p+q=n} \mathrm{Hom}(A_p,B_q)$ with differential

$$(\delta^{H} f)_{p,q} = f_{p-1,q}\,\partial + (-1)^{p+q}\,\delta\, f_{p,q-1},$$

where $f = \{f_{p,q}: A_p \longrightarrow B_q\}$ (**[Hilton-Stammbach]**, p. 169). The same definitions of $\_\otimes\_$ and $\underline{\mathrm{Hom}}(\_,\_)$ are applied to graded modules.

3.1.2 Lemma. Given the dimensionally split short exact sequence of chain complexes of abelian groups (resp. split short exact sequence of graded abelian groups)

$$0 \longrightarrow \mathbb{A}' \xrightarrow{\alpha} \mathbb{A} \xrightarrow{\beta} \mathbb{A}'' \longrightarrow 0,$$

a chain complex $\mathbb{B}$ (resp. a graded abelian group), and a cochain complex $\mathbb{C}$ (resp. a graded abelian group), the sequences

$$0 \longrightarrow \mathbb{A}'\otimes_{\mathbf{Z}}\mathbb{B} \xrightarrow{\alpha\otimes 1} \mathbb{A}\otimes_{\mathbf{Z}}\mathbb{B} \xrightarrow{\beta\otimes 1} \mathbb{A}''\otimes_{\mathbf{Z}}\mathbb{B} \longrightarrow 0, \text{ and}$$

$$0 \longrightarrow \underline{\mathrm{Hom}}_{\mathbf{Z}}(\mathbb{A}'',\mathbb{C}) \xrightarrow{\mathrm{Hom}(\beta,1)} \underline{\mathrm{Hom}}_{\mathbf{Z}}(\mathbb{A},\mathbb{C}) \xrightarrow{\mathrm{Hom}(\alpha,1)} \underline{\mathrm{Hom}}_{\mathbf{Z}}(\mathbb{A}',\mathbb{C}) \longrightarrow 0$$

are dimensionally split short exact sequences of chain complexes.

Proof. This follows quickly from the corresponding statements for modules (**[Spanier]** 5.1.12 & 5.4.7) applied to each grading.

3.1.3 Denote the singular chain functor by $S_{\#}:\mathbf{CGV}^2 \longrightarrow {}_{\mathbf{Z}}\mathbf{Compl}$. If $(X,\emptyset) \in$ ob $\mathbf{CGV}^2$ and $f \in$ morph $\mathbf{CGV}^2$, we write $S_{\#}(X)$ for $S_{\#}(X,\emptyset)$ and $f_{\#}$ for $S_{\#}(f)$.

3.1.4 Remark. If (X,A) is a pair of topological spaces, with the inclusion $i:(A,\emptyset) \subseteq (X,\emptyset)$ and the induced inclusion $j:(X,\emptyset) \subseteq (X,A)$, then we have the induced short exact sequence of singular chain complexes:

$$O \longrightarrow S_*(A) \xrightarrow{i_{\#}} S_*(X) \xrightarrow{j_{\#}} S_*(X, A) \longrightarrow 0,$$

which splits in each dimension p:

$$0 \longrightarrow S_p(A) \underset{z_p}{\overset{i_{\#}}{\rightleftarrows}} S_p(X) \underset{s_p}{\overset{j_{\#}}{\rightleftarrows}} S_p(X,A) \longrightarrow 0,$$

where the quotient module $S_p(X,A)$ is identified with the free submodule of $S_p(X)$ generated by singular p-simplexes of X which do not map into A, and $s_p:S_p(X,A) \subseteq S_p(X)$ denote the inclusion of modules and is an unnatural splitting for j, $j_{\#}\circ s_p = 1$, and $z_p:S_p(X) \longrightarrow S_p(A)$ is the identity on simplexes of X which map into A and zero on the others, $z\circ i_{\#} = 1$. Then

$$s = \{s_p:S_p(X,A) \subseteq S_p(X)\}:S_*(X,A) \longrightarrow S_*(X), \text{ and,}$$

$$z = \{z_p:S_p(X) \longrightarrow S_p(A)\}:S_*(X) \longrightarrow S_*(A)$$

are graded maps. In the diagrams of chain maps graded maps will be denoted by dashed arrows.

3.1.5 Let Δ_n denote the standard n-simplex, that is, $\Delta_n = \{(t_0,\ldots,t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_{i=0}^{n} t_i = 1\}$, and let X and Y be topological spaces. Let $pr_1:Y \underline{x} X \longrightarrow X$ and $pr_2:X \underline{x} Y \longrightarrow Y$ be the canonical projections. If $\omega:\Delta_n \longrightarrow X \underline{x} Y$ is a singular n-simplex, then $\omega = (\sigma \times \eta)\circ d_n$, where $\sigma = pr_1\circ\omega:\Delta_n \longrightarrow X$, $\eta = pr_2\circ\omega:\Delta_n \longrightarrow Y$, and $d_n:\Delta_n \longrightarrow \Delta_n \times \Delta_n$, $x \longrightarrow (x,x)$ is the diagonal map. There is one-to-one correspondence between singular n-chains in the product

$X \underline{x} Y$ and the free $\mathbb{Z}$-module generated by pairs (σ,η) of singular n-simplexes in X and in Y respectively ([MacLane 1] p. 238). Moreover, $\partial\omega = \sum_{i=0}^{n} (-1)^i \omega^{(i)}$ corresponds to $\sum_{i=0}^{n} (-1)^i (\sigma^{(i)},\eta^{(i)})$.

3.2 Coefficient Systems

3.2.1 Definition. A **covariant coefficient system** for J is a covariant functor $\mathbb{M}:J \longrightarrow \mathbf{Compl}_R$ (resp. $\mathbb{M}:J \longrightarrow \mathbf{Ab}$) which is **homotopy invariant**; that is, if $f,g \in J(a,b)$ and f and g are in the same path component of $J(a,b)$, then $\mathbb{M}(f) = \mathbb{M}(g):\mathbb{M}_a \longrightarrow \mathbb{M}_b$. A **contravariant coefficient system** for J is a covariant functor $\mathbb{K}:J^{op} \longrightarrow {}_R\mathbf{CoCompl}$ (resp. $\mathbb{K}:J^{op} \longrightarrow \mathbf{Ab}$) which is homotopy invariant. Note that every covariant or contravariant coefficient system $\mathbb{K}$ taking values in **Ab** can be regarded as a coefficient system taking values in $\mathbf{Compl}_{\mathbb{Z}}$ (resp. ${}_{\mathbb{Z}}\mathbf{CoCompl}$) and concentrated in dimension 0; there is an obvious converse statement.

3.2.2 Definition. Recall that each $J(a,b)$, $a,b \in$ ob J, is a topological space. Impose the discrete topology on the hom sets $\mathbf{Compl}_R(\mathbb{A},\mathbb{B})$ (resp. Ab(A,B)). A covariant coefficient system $\mathbb{M}$ on J is called **continuous** if the functions $\mathbb{M} = \mathbb{M}_{ab}:J(a,b) \longrightarrow \mathbf{Compl}_R(\mathbb{M}_a, \mathbb{M}_b)$, $a,b \in$ ob J, are continuous. Continuous contravariant coefficient systems are defined in similar way.

3.2.3 Lemma. Every continuous coefficient system on J is homotopy invariant.

Proof. This follows easily from the definitions.

3.2.4 Extensions of coefficient systems. Let $\mathbb{M}:J \longrightarrow \mathbf{Compl}_R$ be a homotopy invariant coefficient system on J. We denote by $\pi_0(C)$ a set of path components of a topological space C. Let $a,b \in \text{ob } J$ and $\mathbb{M} = \mathbb{M}_{ab}:J(a,b) \longrightarrow \mathbf{Compl}_R(\mathbb{M}_a, \mathbb{M}_b)$. Since $\mathbb{M}$ is homotopy invariant, $\mathbb{M}_{ab}$ is in fact defined in fact on $\pi_0(J(a,b))$. Let $\mathcal{F}\pi_0(J(a,b))$ be a free $\mathbb{Z}$-module generated by the set $\pi_0(J(a,b))$. Additive structure of chain complexes allows the extension $\mathbb{M}_{ab}$ over $\mathcal{F}\pi_0(J(a,b))$ by the obvious formula. We notice that there is the canonical set map from $S_{\#}(J(a,b))$ to $\pi_0(J(a,b))$, which allows us to extend $\mathbb{M}_{ab}$ over $S_{\#}(J(a,b))$. Explicitly, we extend $\mathbb{M}_{ab}$ over $S_{\#}(J(a,b))$ in the following way.

Let $\eta:\Delta_n \longrightarrow J(a,b)$ be a singular n-simplex and let $f \in \text{image}(\eta)$. We define $\check{\mathbb{M}}(\eta) = \mathbb{M}(f):\mathbb{M}_a \longrightarrow \mathbb{M}_b$. Since Δ_n is path-connected, image(η) lies in one path component of $J(a,b)$ and therefore $\check{\mathbb{M}}(\eta)$ does not depend on a choice of f. Extensions of continuous contravariant coefficient systems are defined in similar way. This type of extension is used in **[Illman]** p.12.

3.3 Singular J-homology

Let $(X,A):J^{op} \longrightarrow \mathbf{CGV}$ be a pair of diagrams over J and let $\mathbb{M}:J \longrightarrow \mathbf{Compl}$ be a coefficient system on J. Our purpose in this section is to construct a chain complex $C^J_*(X,A; \mathbb{M})$ whose homology gives us the "singular" homology of

(X,A) with coefficients in $\mathbb{M}$. The construction will be to some extent similar to that in **[tom Dieck]**, I.11.6 and in **[Vogt 2]**.

We shall develop a singular J-homology in the language of chain complexes composed of ordinary singular simplexes. The chain complexes which we are seeking will be extracted with the help of cokernels of chain maps.

3.3.1 Remark. We begin with an explanation of what is meant by a singular simplex of a diagram X. Let j be an object of J. Modifying Illman's definition of equivariant n-simplexes in **[Illman]**, we call the diagram $Dj \underline{x} \Delta_n$ $(J(\_,j) \underline{x} \Delta_n)$ the **standard singular** J**-n-simplex** of **type** j. Let $X{:}J^{op} \longrightarrow$ CGV be a diagram of topological spaces over J. A natural transformation:

$$T{:}Dj \underline{x} \Delta_n \longrightarrow X$$

is called a **singular** J**-n-simplex** of **type** j in X. We will not further employ the notion of J-simplexes because by the canonical isomorphism $J\text{-CGV}(J(\_,j)\underline{x}\Delta_n, X) \cong \text{CGV}(\Delta_n, Xj)$ (see 1.1.14(2)), a singular J-n-simplex T may be viewed as an ordinary singular n-simplex of the topological space Xj.

3.3.2 Notation. Let $(X,A){:}J^{op} \longrightarrow \text{CGV}^2$ be a saturated J-pair with $i{:}(A,\emptyset) \subseteq (X,\emptyset)$ and the induced inclusion $j{:}(X,\emptyset) \subseteq (X,A)$. Let $a,b \in$ ob J. There are sequences of topological pairs and their maps

$$A_a \xrightarrow{i_a} X_a \xrightarrow{j_a} (X_a,A_a), \text{ and,}$$

$$A_b\underline{x}J(a,b) \xrightarrow{i_b\underline{x}1} X_b\underline{x}J(a,b) \xrightarrow{j_b\underline{x}1} (X_b,A_b)\underline{x}J(a,b),$$

where all maps are inclusions of pairs induced by i and j. An application of the singular functor S gives short exact sequences with the splittings

$$0 \longrightarrow S_*(A_a) \underset{z_a}{\overset{(i_a)_\#}{\rightleftarrows}} S_*(X_a) \underset{s_a}{\overset{(j_a)_\#}{\rightleftarrows}} S_*(X_a,A_a) \longrightarrow 0, \text{ and,}$$

$$0 \longrightarrow S_*(A_b \underline{\times} J(a,b)) \underset{r_{ab}}{\overset{(i_b \underline{\times} 1)_\#}{\rightleftarrows}} S_*(X_b \underline{\times} J(a,b)) \underset{t_{ab}}{\overset{(j_b \underline{\times} 1)_\#}{\rightleftarrows}} S_*((X_b,A_b)\underline{\times} J(a,b)) \longrightarrow 0,$$

where graded maps s_a, z_a, t_{ab}, and r_{ab} are induced canonical splittings (see 3.1.4). Denote by $(\tilde{X},\tilde{A})_{ab\#} = S_\#(\tilde{X}_{ab},\tilde{A}_{ab})$, the chain map induced by the functor S on a pair $(\tilde{X}_{ab},\tilde{A}_{ab}):(X_b,A_b)\underline{\times} J(a,b) \longrightarrow (X_a,A_a)$.

3.3.3 Lemma. Let $a,b \in$ ob J. Each saturated J-pair (X,A) induces the following diagram of chain complexes and graded maps (dashed arrows) and chain maps (solid arrows).

$$\begin{tikzcd}
S_\#(A_b \underline{\times} J(a,b)) \arrow[r, "(\tilde{A}_{ab})_\#"] \arrow[d, shift right, "(i_b \underline{\times} 1)_\#"'] & S_\#(A_a) \arrow[d, shift right, "(i_a)_\#"'] \\
S_\#(X_b \underline{\times} J(a,b)) \arrow[r, "(\tilde{X}_{ab})_\#"] \arrow[u, dashed, shift right, "r_{ab}"'] \arrow[d, shift right, "(j_b \underline{\times} 1)_\#"'] & S_\#(X_a) \arrow[u, dashed, shift right, "z_a"'] \arrow[d, shift right, "(j_a)_\#"'] \\
S_\#((X_b,A_b)\underline{\times} J(a,b)) \arrow[r, "(\tilde{X},\tilde{A})_{ab\#}"] \arrow[u, dashed, shift right, "t_{ab}"'] & S_\#(X_a,A_a) \arrow[u, dashed, shift right, "s_a"']
\end{tikzcd}$$

The diagram of solid arrows commutes in $_{\mathbf{Z}}$**Compl**. The columns made of solid arrows are short exact sequences in $_{\mathbf{Z}}$**Compl** which split (unnaturally) in each dimension. The diagram made of dashed arrows (splittings) and horizontal arrows commute in **Gr**$_{\mathbf{Z}}$**Mod**. The columns made of dashed arrows (splittings) are short exact sequences in **Gr**$_{\mathbf{Z}}$**Mod**.

Proof. Let $a,b \in \mathrm{ob}\ J$. By the naturality of the inclusion $i:(A,\emptyset) \subseteq (X,\emptyset)$ the diagram below commutes:

$$\begin{tikzcd}
A_a \arrow[r, "i_a"] & X_a \arrow[r, "j_a"] & (X_a, A_a) \\
A_b \underline{x} J(a,b) \arrow[u, "\tilde{A}_{ab}"] \arrow[r, "i_a \underline{x} 1"] & X_b \underline{x} J(a,b) \arrow[u, "\tilde{X}_{ab}"'] \arrow[r, "j_b \underline{x} 1"] & (X_b, A_b) \underline{x} J(a,b) \arrow[u, "(\tilde{X}_{ab}, \tilde{A}_{ab})"']
\end{tikzcd}$$

An application of the singular functor S to this diagram proves the commutativity of the claimed diagram with solid arrows in ${}_{\mathbb{Z}}$**Compl.**

Let $\omega \cong (\sigma,\eta)$ be a singular n-simplex of $X_b\underline{x}J(a,b)$ that does not map by $i_b\underline{x}1$ into $A_b\underline{x}J(a,b)$. This is possible only if σ does not map into A_b. Recall, that the pair (X, A) is saturated (see 2.1.3(6)) and therefore $(\tilde{X}_{ab})^{-1}(A_a) = A_b\underline{x}J(a,b)$. If the singular simplex σ does not map into A_b, then $\tilde{X}_{ab}(\omega)$ does not map into A_a. Thus $(\tilde{X}_{ab})_\#$ maps $t_{ab}S_\#\big((X_b,A_b)\underline{x}J(a,b)\big)$ to $s_aS_\#\big((X_a,A_a)\big)$. Therefore the bottom square made of the two bottom dashed arrows and the two bottom horizontal arrows commutes in $\mathbf{Gr_{\mathbb{Z}}Mod}$.

To prove that the upper square made of two upper dashed arrows and two upper horizontal arrows commutes in $\mathbf{Gr_{\mathbb{Z}}Mod}$ we check the commutativity separately on $t_{ab}S_\#\big((X_b,A_b)\underline{x}J(a,b)\big)$ and on its complement $(i_b\underline{x}1)_\#S_\#\big(A_b\underline{x}J(a,b)\big)$. Since $(\tilde{X}_{ab})_\#$ maps $t_{ab}S_\#\big((X_b,A_b)\underline{x}J(a,b)\big)$ to $s_aS_\#\big((X_a,A_a)\big)$ and both r_{ab} and z_a vanish on $t_{ab}S_\#\big((X_b,A_b)\underline{x}J(a,b)\big)$ and $s_aS_\#\big((X_a,A_a)\big)$ respectively, the upper square commutes (taking zero values) in $\mathbf{Gr_{\mathbb{Z}}Mod}$. Since $\tilde{X}_{ab}|A_b\underline{x}J(a,b) = \tilde{A}_{ab}$, and both r_{ab} and z_a are identities on $(i_b\underline{x}1)_\#S_\#\big(A_b\underline{x}J(a,b)\big)$ and $(i_a)_\#S_\#(A_a)$ respectively, the graded maps commute again.

We refer to commutativity of the diagram in 3.3.8 by saying that the diagram containing the solid arrows commutes in **Compl**, and the diagram consisting of the dashed arrows and horizontal solid arrows commutes in **GrMod**.

3.3.4 Definition. Define chain maps $\alpha^{X,A}_{ab}$, a,b $\in$ ob J,

$$S_\#((X_b,A_b)\underline{x}J(a,b)) \otimes_Z M_a \xrightarrow{\alpha^{X,A}_{ab}} S_\#(X_a,A_a) \otimes_Z M_a$$

by $\alpha^{X,A}_{ab} = S_\#(\tilde{X}_{ab},\tilde{A}_{ab}) \otimes 1_{M_a}$. Clearly each $\alpha^{X,A}_{ab}$ is a chain map. As special cases, we have α^{X}_{ab} and α^{A}_{ab}, which are defined for pairs $(X,\emptyset)$ and $(A,\emptyset)$ respectively,

$$S_\#(X_b\underline{x}J(a,b)) \otimes_Z M_a \xrightarrow{\alpha^{X}_{ab}} S_\#(X_a) \otimes_Z M_a$$

$$S_\#(A_b\underline{x}J(a,b)) \otimes_Z M_a \xrightarrow{\alpha^{A}_{ab}} S_\#(A_a) \otimes_Z M_a$$

defined by $\alpha^{X}_{ab} = (\tilde{X}_{ab})_\# \otimes 1_{M_a}$ and $\alpha^{A}_{ab} = (\tilde{A}_{ab})_\# \otimes 1_{M_a}$.

3.3.5 Proposition. Let a,b $\in$ ob J. Each saturated J–pair (X,A) induces the following diagram of chain complexes and graded maps (dashed arrows) and chain maps (solid arrows). The diagram of solid arrows commutes in $\mathbf{Compl}_R$. The columns made of solid arrows are short exact sequences in $\mathbf{Compl}_R$, splitting (unnaturally) in each dimension. The diagram made of dashed arrows (splittings) and horizontal arrows commute in $\mathbf{GrMod}_R$. The columns made of dashed arrows (splittings) are short exact sequences in $\mathbf{GrMod}_R$.

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\mathcal{S}_{\#}(A_b \underline{\times} \mathcal{J}(a,b)) \otimes \mathbb{M}_a \arrow[r, "\alpha^{A}_{ab}"] \arrow[d, shift right, "(i_b\underline{\times}1)_{\#}\otimes 1"'] & \mathcal{S}_{\#}(A_a) \otimes \mathbb{M}_a \arrow[d, shift right, "(i_a)_{\#}\otimes 1"'] \\
\mathcal{S}_{\#}(X_b \underline{\times} \mathcal{J}(a,b)) \otimes \mathbb{M}_a \arrow[r, "\alpha^{X}_{ab}"] \arrow[u, dashed, shift right, "r_{ab}\otimes 1"'] \arrow[d, shift right, "(j_b\underline{\times}1)_{\#}\otimes 1"'] & \mathcal{S}_{\#}(X_a) \otimes \mathbb{M}_a \arrow[u, dashed, shift right, "z_a\otimes 1"'] \arrow[d, shift right, "(j_a)_{\#}\otimes 1"'] \\
\mathcal{S}_{\#}((X_b, A_b) \underline{\times} \mathcal{J}(a,b)) \otimes \mathbb{M}_a \arrow[r, "\alpha^{X,A}_{ab}"'] \arrow[u, dashed, shift right, "t_{ab}\otimes 1"'] & \mathcal{S}_{\#}(X_a, A_a) \otimes \mathbb{M}_a \arrow[u, dashed, shift right, "s_a\otimes 1"']
\end{tikzcd}
$$

Proof. Tensoring the diagram 3.3.3 with $\mathbb{M}_a$ gives the commutativity for solid and dashed arrows. Since columns of the diagram in 3.3.3 are split both for solid and dashed arrows, Lemma 3.1.2 and definition 3.3.4 yield the claim.

3.3.6 Definition. Recall that we identify singular simplexes ω of the product $X_b\underline{\times}\mathcal{J}(a,b)$ with pairs (σ,η) (see 3.1.5). Define maps $\beta^{X,A}_{ab}$

$$\mathcal{S}_{\#}((X_b,A_b)\underline{\times}\mathcal{J}(a,b)) \otimes_{\mathbb{Z}} \mathbb{M}_b \xrightarrow{\beta^{X,A}_{ab}} \mathcal{S}_{\#}(X_b,A_b) \otimes_{\mathbb{Z}} \mathbb{M}_b$$

by $\beta^{X,A}_{ab}([\sigma,\eta]\otimes m) = [\sigma]\otimes\check{\mathbb{M}}(\eta)(m)$, $a,b \in \operatorname{ob} \mathcal{J}$. It is not obvious that each $\beta^{X,A}_{ab}$ is a chain map. As special cases, we have β^{X}_{ab} and β^{A}_{ab}, which are defined for pairs $(X,\varnothing)$ and $(A,\varnothing)$ respectively,

$$\mathcal{S}_{\#}(X_b\underline{\times}\mathcal{J}(a,b)) \otimes_{\mathbb{Z}} \mathbb{M}_b \xrightarrow{\beta^{X}_{ab}} \mathcal{S}_{\#}(X_b) \otimes_{\mathbb{Z}} \mathbb{M}_b, \text{ and}$$

$$\mathcal{S}_{\#}(A_b\underline{\times}\mathcal{J}(a,b)) \otimes_{\mathbb{Z}} \mathbb{M}_b \xrightarrow{\beta^{A}_{ab}} \mathcal{S}_{\#}(A_b) \otimes_{\mathbb{Z}} \mathbb{M}_b.$$

3.3.7 Lemma. Each $\beta^{X,A}_{ab}$ is a chain map.

Proof. If $[\sigma,\eta]\otimes m$ is a generator of $S_*\big((X_b,A_b)\underline{x}J(a,b)\big)\otimes_{\mathbb{Z}}\mathcal{M}_b$, then

$$\partial\big([\sigma,\eta]\otimes m\big) \;=\; \partial([\sigma,\eta])\otimes m \;+\; (-1)^{|\sigma|}[\sigma,\eta]\otimes\partial m$$

$$= \sum_{i=0}^{|\sigma|}(-1)^i\,[\sigma^{(i)},\eta^{(i)}]\otimes m + (-1)^{|\sigma|}[\sigma,\eta]\otimes\partial m.$$

Since $\text{image}(\eta^{(k)}) \subseteq \text{image}(\eta)$, $\check{\mathcal{M}}(\eta^{(k)}) = \check{\mathcal{M}}(\eta)$, and

$$\beta^{X,A}_{ab}\partial\big(([\sigma],\eta)\otimes m\big) = \sum_{i=0}^{|\sigma|}(-1)^i\,[\sigma^{(i)}]\otimes\check{\mathcal{M}}(\eta^{(i)})(m) + (-1)^{|\sigma|}[\sigma]\otimes\check{\mathcal{M}}(\eta)(\partial m)$$

$$= \sum_{i=0}^{|\sigma|}(-1)^i[\sigma^{(i)}]\otimes\check{\mathcal{M}}(\eta)(m) + (-1)^{|\sigma|}[\sigma]\otimes\check{\mathcal{M}}(\eta)(\partial m).$$

On the other hand,

$$\partial\beta^{X,A}_{ab}\big([\sigma,\eta]\otimes m\big) = \partial\big([\sigma]\otimes\check{\mathcal{M}}(\eta)(m)\big) = \partial[\sigma]\otimes\check{\mathcal{M}}(\eta)(m) + (-1)^{|\sigma|}[\sigma]\otimes\partial\big(\check{\mathcal{M}}(\eta)(m)\big)$$

$$= \sum_{i=0}^{|\sigma|}(-1)^i[\sigma^{(i)}]\otimes\check{\mathcal{M}}(\eta)(m) + (-1)^{|\sigma|}[\sigma]\otimes\partial\big(\check{\mathcal{M}}(\eta)(m)\big).$$

Note that $\check{\mathcal{M}}(\eta)$ is a chain map, so $\partial\big(\check{\mathcal{M}}(\eta)(m)\big) = \check{\mathcal{M}}(\eta)(\partial m)$. Therefore $\beta^{X,A}_{ab}\partial\big([\sigma,\eta]\otimes m\big) = \partial\beta^{X,A}_{ab}\big([\sigma,\eta]\otimes m\big)$.

3.3.8 Lemma. Let $a,b \in \text{ob}\, J$. Each saturated J-pair (X,A) induces the following diagram of chain complexes and graded maps (dashed arrows) and chain maps (solid arrows). The diagram of solid arrows commutes in $\mathbf{Compl}_R$. The columns made of solid arrows are short exact sequences in $\mathbf{Compl}_R$, splitting (unnaturally) in each dimension. The diagram made of dashed arrows and horizontal arrows commute in $\mathbf{GrMod}_R$. The columns made of dashed arrows (splittings) are short exact sequences in $\mathbf{GrMod}_R$.

$$\begin{tikzcd}
S_\#(A_b \underline{x} J(a,b))\otimes \mathbb{M}_a \arrow[r, "\beta^A_{ab}"] \arrow[d, shift right, "(i_b\underline{x}1)_\#\otimes 1"'] & S_\#(A_b)\otimes \mathbb{M}_b \arrow[d, shift right, "(i_b)_\#\otimes 1"'] \\
S_\#(X_b \underline{x} J(a,b))\otimes \mathbb{M}_a \arrow[u, dashed, shift right, "r_{ab}\otimes 1"'] \arrow[r, "\beta^X_{ab}"] \arrow[d, shift right, "(j_b\underline{x}1)_\#\otimes 1"'] & S_\#(X_b)\otimes \mathbb{M}_b \arrow[u, dashed, shift right, "z_b\otimes 1"'] \arrow[d, shift right, "(j_b)_\#\otimes 1"'] \\
S_\#((X_b, A_b)\underline{x} J(a,b))\otimes \mathbb{M}_a \arrow[u, dashed, shift right, "t_{ab}\otimes 1"'] \arrow[r, "\beta^{X,A}_{ab}"'] & S_\#(X_b, A_b)\otimes \mathbb{M}_b \arrow[u, dashed, shift right, "s_b\otimes 1"']
\end{tikzcd}$$

Proof. By checking directly we see that the square inside the following diagram of solid arrows commutes.

$$\begin{tikzcd}
S_\#(A_b \underline{x} J(a,b))\otimes \mathbb{M}_a \arrow[r, "\beta^A_{ab}"] \arrow[d, "(i_b\underline{x}1)_\#\otimes 1"'] & S_\#(A_b)\otimes \mathbb{M}_b \arrow[d, "(i_b)_\#\otimes 1"] & (\sigma,\eta)\otimes m \arrow[r] \arrow[d] & \sigma\otimes\check{\mathbb{M}}(\eta)(m) \arrow[d] \\
S_\#(X_b \underline{x} J(a,b))\otimes \mathbb{M}_a \arrow[r, "\beta^X_{ab}"] & S_\#(X_b)\otimes \mathbb{M}_b & (i_b\circ\sigma,\eta)\otimes m \arrow[r] & i_b\circ\sigma\otimes\check{\mathbb{M}}(\eta)(m)
\end{tikzcd}$$

3.3.9 Definition. Denote by ι_a, $a \in$ ob J, the canonical injection of $S_\#(X_a,A_a)\otimes \mathbb{M}_a$ into $\sum_c S_\#(X_c,A_c)\otimes \mathbb{M}_c$. We define chain maps $\alpha^{X,A} = \sum_{a,b} \iota_a\alpha^{X,A}_{ab}$ and $\beta^{X,A} = \sum_{a,b} \iota_b\beta^{X,A}_{ab}$

$$\sum_{a,b} S_\#((X_b,A_b)\underline{x}J(a,b)) \otimes_Z \mathbb{M}_a \underset{\beta^{X,A}}{\overset{\alpha^{X,A}}{\rightrightarrows}} \sum_c S_\#(X_c,A_c) \otimes_Z \mathbb{M}_c,$$

and finally define $\Phi^{X,A} = \alpha^{X,A} - \beta^{X,A}$. As special cases, we have α^X, α^A, β^X, β^A, and $\Phi^X = \alpha^X - \beta^X$, $\Phi^A = \alpha^A - \beta^A$

$$\sum_{a,b} S_\#(X_b \underline{x} J(a,b)) \otimes_Z \mathbb{M}_a \underset{\beta^X}{\overset{\alpha^X}{\rightrightarrows}} \sum_{c} S_\#(X_c) \otimes_Z \mathbb{M}_c,$$

$$\sum_{a,b} S_\#(X_b \underline{x} J(a,b)) \otimes_Z \mathbb{M}_a \underset{\beta^A}{\overset{\alpha^A}{\rightrightarrows}} \sum_{c} S_\#(X_c) \otimes_Z \mathbb{M}_c.$$

3.3.10 Definition. Define the J-singular chain complex of (X,A) with coefficients in $\mathbb{M}$ as

$$C_*^J(X,A;\ \mathbb{M}) = \text{cokernel}(\Phi^{X,A}),$$

and, as special cases, $C_*^J(X;\ \mathbb{M}) = \text{cokernel}(\Phi^X)$ for $(X,\varnothing)$ and $C_*^J(A;\ \mathbb{M}) = \text{cokernel}(\Phi^A)$ for $(A,\varnothing)$.

3.3.11 Proposition. Let a,b $\in$ ob J. Each saturated J-pair (X,A) induces the following diagram of chain complexes and graded maps (dashed arrows) and chain maps (solid arrows).

$$\begin{tikzcd}
\sum_{a,b} S_\#(A_b \underline{x} J(a,b)) \otimes \mathbb{M}_a \arrow[r, shift left, "\alpha^A"] \arrow[r, shift right, "\beta^A"'] \arrow[d, shift right, "((i_b \underline{x} 1)_\# \otimes 1)"'] & \sum_{c} S_\#(A_c) \otimes \mathbb{M}_c \arrow[d, shift right, "((i_c)_\# \otimes 1)"'] \\
\sum_{a,b} S_\#(X_b \underline{x} J(a,b)) \otimes \mathbb{M}_a \arrow[u, dashed, shift right, "(r_{a\,b} \otimes 1)"'] \arrow[r, shift left, "\alpha^X"] \arrow[r, shift right, "\beta^X"'] \arrow[d, shift right, "((j_b \underline{x} 1)_\# \otimes 1)"'] & \sum_{c} S_\#(X_c) \otimes \mathbb{M}_c \arrow[u, dashed, shift right, "(z_c \otimes 1)"'] \arrow[d, shift right, "((j_c)_\# \otimes 1)"'] \\
\sum_{a,b} S_\#((X_b, A_b) \underline{x} J(a,b)) \otimes \mathbb{M}_a \arrow[u, dashed, shift right, "(t_{a\,b} \otimes 1)"'] \arrow[r, shift left, "\alpha^{X,A}"] \arrow[r, shift right, "\beta^{X,A}"'] & \sum_{c} S_\#(X_c, A_c) \otimes \mathbb{M}_c \arrow[u, dashed, shift right, "(s_c \otimes 1)"']
\end{tikzcd}$$

The diagram of solid arrows commutes in $\mathbf{Compl}_R$. The columns made of solid arrows are short exact sequences in $\mathbf{Compl}_R$, splitting (unnaturally) in each dimension. The diagram made of dashed arrows (splittings) and horizontal arrows commute in $\mathbf{GrMod}_R$. The columns made of dashed arrows (splittings) are short exact sequences in $\mathbf{GrMod}_R$.

Proof. Direct sums preserve exactness and splittings. Commutativity follows from that of the components of α's and β's as given in 3.3.5 and 3.3.8.

3.3.12 Commutativity of the diagram in 3.3.11 induces one chain map, $i_\#$, between cokernels of Φ^A and Φ^X, and another, $j_\#$, between cokernels of Φ^X and $\Phi^{X,A}$.

3.3.13 Corollary

(1) $0 \longrightarrow C^J_*(A;\, \mathbb{M}) \xrightarrow{i_\#} C^J_*(X;\, \mathbb{M}) \xrightarrow{j_\#} C^J_*(X,A;\, \mathbb{M}) \longrightarrow 0$

is a s.e.s. with unnatural splitting.

(2) $C^J_*(X,A;\, \mathbb{M}) \cong C^J_*(X;\, \mathbb{M}) \,/\, C^J_*(A;\, \mathbb{M})$.

Proof. We take the cokernel sequence of rows of the diagram in 3.3.11. Splittings assure that the cokernel sequence is exact and unnaturally splits

3.3.14 Definition. We define the **singular *J*-homology** of (X,A) with coefficients in $\mathbb{M}$ as

$$H^J_*(X,A;\, \mathbb{M}) = H_*(C^J_*(X,A;\, \mathbb{M}))$$

where H_* is the homology functor. The boundary operator ∂^J_* of the singular *J*-homology theory is defined to be the connecting morphism of a long exact sequence in homology of the short exact sequence of chain complexes 3.3.13(1).

From Corollary 3.3.13 and 3.3.14 we have.

3.3.15 Proposition (EXACTNESS). The singular J-homology theory satisfies the Exactness Axiom.

3.4 Singular J-cohomology

Let $(X,A):J^{op} \longrightarrow$ CGV be a pair of diagrams over J and let $\mathbb{K}:J^{op} \longrightarrow {}_R\mathbf{CoCompl}$ be a contravariant coefficient system on J. Our purpose in this section is to construct a cochain complex $C^*_J(X,A;\ \mathbb{K})$ whose homology gives us the "singular" cohomology of (X,A) with coefficients in $\mathbb{K}$. We shall develop a singular J-cohomology by the technique of cochain complexes made of ordinary singular simplexes (and $\mathbb{K}$), and the cochain complexes which we are seeking will be extracted with the help of kernels of cochain maps.

3.4.1 Definition. Recall $(\tilde{X}_{ab},\tilde{A}_{ab})_{\#}:S_{\#}((X_b,A_b)\underline{x}J(a,b)) \longrightarrow S_{\#}(X_a,A_a)$.
Define chain maps $\phi^{X,A}_{ab}$, a,b $\in$ ob J,

$$\phi^{X}_{ab}:\underline{\mathrm{Hom}}_Z(S_{\#}(X_a,A_a),\ \mathbb{K}_a) \xrightarrow{\phi^{X,A}_{ab}} \underline{\mathrm{Hom}}_Z(S_{\#}((X_b,A_b)\underline{x}J(a,b)),\ \mathbb{K}_a)$$

by the assignment $\phi^{X,A}_{ab}(f) = \{f_{p,q}\circ(\tilde{X}_{ab},\tilde{A}_{ab})_{\#}:S_p((X_b,A_b)\underline{x}J(a,b)) \longrightarrow \mathbb{K}_{a_q}\}$ for $f = \{f_{p,q}:S_p(X_a,A_a) \longrightarrow \mathbb{K}_{a_q}\}$. As a special case, we have ϕ^X_{ab} and ϕ^A_{ab}, which are defined for pairs $(X,\emptyset)$ and $(A,\emptyset)$ respectively, with

$$\phi^X_{ab}:\underline{\mathrm{Hom}}(S_{\#}(X_a),\ \mathbb{K}_a) \longrightarrow \underline{\mathrm{Hom}}(S_{\#}(X_b\underline{x}J(a,b)),\ \mathbb{K}_a) \text{ and}$$

$$\phi^A_{ab}:\underline{\mathrm{Hom}}(S_\#(A_a),\ \mathbb{K}_a) \longrightarrow \underline{\mathrm{Hom}}(S_\#(A_b \underline{x} J(a,b)),\ \mathbb{K}_a).$$

3.4.2 Lemma. Each ϕ^X_{ab} is a chain map.

Proof. This easily follows since $(\tilde{X}_{ab})_\#$ is a chain map.

3.4.3 Lemma. Let $a,b \in \mathrm{ob}\ J$. Each saturated J-pair (X,A) induces the following diagram of cochain complexes and graded maps (dashed arrows) and cochain maps (solid arrows).

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(S_\#(A_a),\ \mathbb{K}_a) \arrow[r, "\phi^A_{ab}"] \arrow[d, dashed, shift left, "\underline{\mathrm{Hom}}(z_a{,}1)"] & \underline{\mathrm{Hom}}(S_\#(A_b\ \underline{x}\ J(a,b)),\ \mathbb{K}_a) \arrow[d, dashed, shift left, "\underline{\mathrm{Hom}}(r_{ab}{,}1)"] \\
\underline{\mathrm{Hom}}(S_\#(X_a),\ \mathbb{K}_a) \arrow[r, "\phi^X_{ab}"] \arrow[u, shift left, "\underline{\mathrm{Hom}}((i_a)_\#{,}1)"] \arrow[d, dashed, shift left, "\underline{\mathrm{Hom}}(s_a{,}1)"] & \underline{\mathrm{Hom}}(S_\#(X_b\ \underline{x}\ J(a,b)),\ \mathbb{K}_a) \arrow[u, shift left, "\underline{\mathrm{Hom}}((i_b \underline{x} 1)_\#{,}1)"] \arrow[d, dashed, shift left, "\underline{\mathrm{Hom}}(t_{ab}{,}1)"] \\
\underline{\mathrm{Hom}}(S_\#(X_a, A_a),\ \mathbb{K}_a) \arrow[r, "\phi^{X,A}_{ab}"'] \arrow[u, shift left, "\underline{\mathrm{Hom}}((j_a)_\#{,}1)"] & \underline{\mathrm{Hom}}(S_\#((X_b, A_b)\ \underline{x}\ J(a,b)),\ \mathbb{K}_a) \arrow[u, shift left, "\underline{\mathrm{Hom}}((j_b \underline{x} 1)_\#{,}1)"]
\end{tikzcd}$$

The diagram of solid arrows commutes in ${}_R\mathbf{CoCompl}$. The columns made of solid arrows are short exact sequences in ${}_R\mathbf{CoCompl}$, splitting (unnaturally) in each dimension. The diagram made of dashed arrows (splittings) and horizontal arrows commute in $\mathbf{Gr}_R\mathbf{Mod}$. The columns made of dashed arrows (splittings) are short exact sequences in $\mathbf{Gr}_R\mathbf{Mod}$.

Proof. Apply $\underline{\mathrm{Hom}}_Z(\_,\ \mathbb{K}_a)$ to the diagram in 3.3.3 to get commutativity. Since columns of the diagram in 3.3.3 are split both for solid and dashed arrows, Lemma 3.1.2 yields the exactness.

3.4.4 Definition. Recall that we identify singular simplexes ω of the product $X_b \times \underline{J}(a,b)$ with pairs (σ,η) (see 3.1.4). Define chain maps $\psi^{X,A}_{ab}$, $a,b \in \text{ob } J$,

$$\underline{\text{Hom}}_{\mathbb{Z}}(S_\#(X_b,A_b),\ \kappa_b) \xrightarrow{\psi^{X,A}_{ab}} \underline{\text{Hom}}_{\mathbb{Z}}(S_\#((X_b,A_b)\times\underline{J}(a,b)),\ \kappa_a),$$

by the assignment $\psi^{X,A}_{ab}(f)_{p,q}([\sigma,\eta]) = \check{\kappa}(\eta)(f_{p,q}([\sigma])$ for $f = \{f_{p,q}:S_p(X_b,A_b) \longrightarrow {\kappa_b}_q\}$. It is not obvious that each $\psi^{X,A}_{ab}$ is a chain map. As special cases, we have ψ^{X}_{ab} and ψ^{A}_{ab}, which are defined for pairs $(X,\varnothing)$ and $(A,\varnothing)$ respectively, with

$$\underline{\text{Hom}}_{\mathbb{Z}}(S_\#(X_b),\ \kappa_b) \xrightarrow{\psi^{X}_{ab}} \underline{\text{Hom}}_{\mathbb{Z}}(S_\#(X_b\times\underline{J}(a,b)),\ \kappa_a) \quad \text{and}$$

$$\underline{\text{Hom}}_{\mathbb{Z}}(S_\#(A_b),\ \kappa_b) \xrightarrow{\psi^{A}_{ab}} \underline{\text{Hom}}_{\mathbb{Z}}(S_\#(A_b\times\underline{J}(a,b)),\ \kappa_a).$$

3.4.5 Lemma. Each $\psi^{X,A}_{ab}$ is a chain map.

Proof. Recall that $(\delta^H f)_{p,q} = f_{p-1,q}\partial + (-1)^{p+q}\ \delta\ f_{p,q-1}$, where $f = \{f_{p,q}:S_p(X_b,A_b) \longrightarrow {\kappa_b}_q\}$, each $S_\#(X_b,A_b)$ is a chain complex with the differential ∂, and each κ_a is a cochain complex with differential δ. Then

$$\delta^H\ (\psi^{X,A}_{ab}(f))_{p,q} = \psi^{X,A}_{ab}(f)_{p-1,q}\partial + (-1)^{p+q}\ \delta\ \psi^{X,A}_{ab}(f)_{p,q-1},$$

$$\partial[\sigma] = \sum_{k=0}^{n}(-1)^k\ [\sigma^{(k)}], \text{ and } \partial([\sigma,\eta]) = \sum_{k=0}^{n}(-1)^k\ [\sigma^{(k)},\eta^{(k)}].$$

Therefore $\delta^H\ (\psi^{X,A}_{ab}(f))_{p,q}([\sigma,\eta]) =$

$\psi^{X,A}_{ab}(f)_{p-1,q}\partial([\sigma,\eta]) + (-1)^{p+q}\ \delta\ \psi^{X,A}_{ab}(f)_{p,q-1}([\sigma,\eta]) =$

$\sum_{k=0}^{n} (-1)^k\ \check{\mathbb{K}}(\eta^{(k)})(f_{p-1,q}([\sigma^{(k)}])) + (-1)^{p+q}\ \delta\ \check{\mathbb{K}}(\eta)(f_{p,q-1}([\sigma]))$.

On the other hand $\psi^{X,A}_{ab}(\delta^H\ f)_{p,q}([\sigma,\eta]) = \check{\mathbb{K}}(\eta)((\delta^H f)_{p,q}([\sigma])) = \check{\mathbb{K}}(\eta)(f_{p-1,q}\partial[\sigma] + (-1)^{p+q}\ \delta\ f_{p,q-1}([\sigma])) = \sum_{k=0}^{n} (-1)^k\ \check{\mathbb{K}}(\eta)(f_{p-1,q}([\sigma^{(k)}])) +$ $(-1)^{p+q}\ \check{\mathbb{K}}(\eta)\ \delta\ (f_{p-1,q}([\sigma]))$.

Note that $\check{\mathbb{K}}(\eta)$ is a cochain map: $\check{\mathbb{K}}(\eta)\delta = \delta\check{\mathbb{K}}(\eta)$, and $\check{\mathbb{K}}(\eta) = \check{\mathbb{K}}(\eta^{(k)})$ since image$(\eta^{(k)}) \subseteq$ image(η). Therefore $\delta^H(\psi^{X,A}_{ab}(f))_{p,q}([\sigma,\eta]) = \psi^{X,A}_{ab}(\delta^H f)_{p,q}([\sigma,\eta])$.

3.4.6 Lemma. Let $a,b \in$ ob J. Each saturated J–pair (X,A) induces the following diagram of cochain complexes and graded maps (dashed arrows) and cochain maps (solid arrows).

$$\begin{tikzcd}[column sep=huge, row sep=large]
\underline{\mathrm{Hom}}(S_\#(A_b),\ \mathbb{K}_b) \arrow[r, "\psi^A_{ab}"] \arrow[d, dashed, shift left, "\underline{\mathrm{Hom}}(z_b{,}1)"] & \underline{\mathrm{Hom}}(S_\#(A_b \underline{x} J(a,b)),\ \mathbb{K}_a) \arrow[d, dashed, shift left, "\underline{\mathrm{Hom}}(r_{ab}{,}1)"] \\
\underline{\mathrm{Hom}}(S_\#(X_b),\ \mathbb{K}_b) \arrow[r, "\psi^X_{ab}"] \arrow[u, shift left, "\underline{\mathrm{Hom}}((i_b)_\#{,}1)"] \arrow[d, dashed, shift left, "\underline{\mathrm{Hom}}(s_b{,}1)"] & \underline{\mathrm{Hom}}(S_\#(X_b \underline{x} J(a,b)),\ \mathbb{K}_a) \arrow[u, shift left, "\underline{\mathrm{Hom}}((i_b \underline{x} 1)_\#{,}1)"] \arrow[d, dashed, shift left, "\underline{\mathrm{Hom}}(t_{ab}{,}1)"] \\
\underline{\mathrm{Hom}}(S_\#(X_b, A_b),\ \mathbb{K}_b) \arrow[r, "\psi^{X,A}_{ab}"'] \arrow[u, shift left, "\underline{\mathrm{Hom}}((j_b)_\#{,}1)"] & \underline{\mathrm{Hom}}(S_\#((X_b, A_b) \underline{x} J(a,b)),\ \mathbb{K}_a) \arrow[u, shift left, "\underline{\mathrm{Hom}}((j_b \underline{x} 1)_\#{,}1)"]
\end{tikzcd}$$

The diagram of solid arrows commutes in ${}_R$**CoCompl**. The columns made of solid arrows are short exact sequences in ${}_R$**CoCompl**, splitting (unnaturally) in each dimension. The diagram made of dashed arrows (splittings) and horizontal arrows commute in $\mathbf{Gr}_R\mathbf{Mod}$. The columns made of dashed arrows (splittings) are short exact sequences in $\mathbf{Gr}_R\mathbf{Mod}$.

Proof. We shall show the commutativity of the upper square of solid arrows.

Let $f = \{f_{p,q}: S_p(X_b) \longrightarrow \mathbb{K}_{b_q}\}$. Then $\left[\psi^A_{ab}(\underline{\mathrm{Hom}}((i_b)_\#, 1_{\mathbb{K}_b})(f))\right]_{p,q}(\sigma,\eta) = \check{\mathbb{K}}(\eta)\left[(\underline{\mathrm{Hom}}((i_b)_\#, 1_{\mathbb{K}_b})(f))_{p,q}(\sigma)\right] = \check{\mathbb{K}}(\eta)\left[f_{p,q}((i_b)_\#(\sigma))\right]$ and

$\left[\underline{\mathrm{Hom}}((i_b\underline{x}1_{J(a,b)})_\#, 1_{\mathbb{K}_b})(\psi^X_{ab}(f))\right]_{p,q}(\sigma,\eta) = \left[\psi^X_{ab}(f)\right]_{p,q}((i_b\underline{x}1_{J(a,b)})_\#(\sigma,\eta)) = \left[\psi^X_{ab}(f)\right]_{p,q}((i_b)_\#(\sigma),\eta) = \check{\mathbb{K}}(\eta)\left[f_{p,q}((i_b)_\#(\sigma))\right]$.

Therefore $\psi^A_{ab} \circ \underline{\mathrm{Hom}}((i_b)_\#, 1_{\mathbb{K}_b}) = \underline{\mathrm{Hom}}((i_b\underline{x}1_{J(a,b)})_\#, 1_{\mathbb{K}_b}) \circ \psi^X_{ab}$.

3.4.7 Definition. Denote by p_e, $e \in \mathrm{ob}\,J$, the canonical projections from $\prod_a \underline{\mathrm{Hom}}(S_\#(X_a), \mathbb{K}_a)$ to $\underline{\mathrm{Hom}}(S_\#(X_e), \mathbb{K}_e)$. Define chain maps

$$\phi^{X,A} = \prod_{a,b} \phi^{X,A}_{ab} p_a \quad \text{and} \quad \psi^{X,A} = \prod_{a,b} \psi^{X,A}_{ab} p_b, \text{ where}$$

$$\prod_a \underline{\mathrm{Hom}}(S_\#(X_a,A_a), \mathbb{K}_a) \underset{\psi^{X,A}}{\overset{\phi^{X,A}}{\rightrightarrows}} \prod_{b,c} \underline{\mathrm{Hom}}(S_\#((X_c,A_c)\underline{x}J(b,c)), \mathbb{K}_b),$$

and finally we define $\Psi^{X,A} = \phi^{X,A} - \psi^{X,A}$. As a special case, we have Ψ^X and Ψ^A, which are defined for pairs $(X,\varnothing)$ and $(A,\varnothing)$ respectively, with

$$\prod_a \underline{\mathrm{Hom}}(S_\#(X_a), \mathbb{K}_a) \underset{\psi^{X}}{\overset{\phi^{X}}{\rightrightarrows}} \prod_{b,c} \underline{\mathrm{Hom}}(S_\#(X_c\underline{x}J(b,c)), \mathbb{K}_b) \text{ and}$$

$$\prod_a \underline{\mathrm{Hom}}(S_\#(A_a), \mathbb{K}_a) \underset{\psi^A}{\overset{\phi^A}{\rightrightarrows}} \prod_{b,c} \underline{\mathrm{Hom}}(S_\#(A_c \underline{x} J(b,c)), \mathbb{K}_b).$$

3.4.8 Proposition. Let $a,b \in \mathrm{ob}\, J$. Each saturated J-pair (X,A) induces the following diagram of cochain complexes and graded maps (dashed arrows) and cochain maps (solid arrows).

$$\begin{array}{ccccc}
\prod_a \underline{\mathrm{Hom}}(S_\#(A_a), \mathbb{K}_a) & \underset{\psi^A}{\overset{\phi^A}{\rightrightarrows}} & \prod_{b,c} \underline{\mathrm{Hom}}(S_\#(A_c \underline{x} J(b,c)), \mathbb{K}_b) \\
(\underline{\mathrm{Hom}}((i_a)_\#, 1)) \uparrow \downarrow (\underline{\mathrm{Hom}}(z_a, 1)) & & (\underline{\mathrm{Hom}}((i_c \underline{x} 1)_\#, 1)) \uparrow \downarrow (\underline{\mathrm{Hom}}(r_{bc}, 1)) \\
\prod_a \underline{\mathrm{Hom}}(S_\#(X_a), \mathbb{K}_a) & \underset{\psi^X}{\overset{\phi^X}{\rightrightarrows}} & \prod_{b,c} \underline{\mathrm{Hom}}(S_\#(A_c \underline{x} J(b,c)), \mathbb{K}_b) \\
(\underline{\mathrm{Hom}}((j_a)_\#, 1)) \uparrow \downarrow (\underline{\mathrm{Hom}}(s_a, 1)) & & (\underline{\mathrm{Hom}}((j_c \underline{x} 1)_\#, 1)) \uparrow \downarrow (\underline{\mathrm{Hom}}(t_{bc}, 1)) \\
\prod_a \underline{\mathrm{Hom}}(S_\#(X_a, A_a), \mathbb{K}_a) & \underset{\psi^{X,A}}{\overset{\phi^{X,A}}{\rightrightarrows}} & \prod_{b,c} \underline{\mathrm{Hom}}(S_\#((X_c, A_c) \underline{x} J(b,c)), \mathbb{K}_b)
\end{array}$$

The diagram of solid arrows commutes in ${}_R$**CoCompl**. The columns made of solid arrows are short exact sequences in ${}_R$**CoCompl**, splitting (unnaturally) in each dimension. The diagram made of dashed arrows (splittings) and horizontal arrows commute in $\mathbf{Gr}_R\mathbf{Mod}$. The columns made of dashed arrows (splittings) are short exact sequences in $\mathbf{Gr}_R\mathbf{Mod}$.

Proof. Products preserve exactness and splittings. Commutativity follows from that of components of ϕ's and ψ's as given in 3.4.3 and 3.4.6.

3.4.9 Definition. Finally we define the J-singular cohomology of (X,A) with coefficients in $\mathbb{K}$ as

$$C^*_J(X,A;\ \mathbb{K}) = \mathrm{kernel}(\Psi^{X,A}),$$

and as special cases $C^*_J(X;\ \mathbb{K}) = \mathrm{kernel}(\Psi^X)$ for $(x,\emptyset)$, $C^*_J(A;\ \mathbb{K}) = \mathrm{kernel}(\Psi^A)$ for $(A,\emptyset)$.

Commutativity of the diagram in 3.4.8 induces one chain maps $j^\#$ between kernels of $\Psi^{X,A}$ and Ψ^X, and another $i^\#$ between kernels of Ψ^X and Ψ^A.

3.4.10 Corollary

(1) $0 \longrightarrow C^*_J(X,A;\ \mathbb{M}) \xrightarrow{j^\#} C^*_J(X;\ \mathbb{M}) \xrightarrow{i^\#} C^*_J(A;\ \mathbb{M}) \longrightarrow 0$

is a s.e.s. which splits in each dimension.

(2) $C^*_J(X,A;\ \mathbb{M}) \cong C^*_J(X;\ \mathbb{M}) \,/\, C^*_J(A;\ \mathbb{M})$.

Proof. We take the kernel sequence of rows of the diagram in 3.4.8. Splittings assure that the kernel sequence is exact and unnaturally splits.

3.4.11 Definition. We define the **singular *J*-cohomology** of (X,A) with coefficients in $\mathbb{K}$ as

$$H^*_J(X,A;\ \mathbb{K}) = H_*(C^*_J(X,A;\ \mathbb{K}))$$

where H_* is the homology functor. The coboundary operator $\delta^*_J = \delta^*_J(X,A;\ \mathbb{K})$ of the singular J-cohomology theory is defined to be the connecting morphism of long exact sequence in cohomology of the short exact sequence of cochain complexes in 3.4.10(1).

From 3.4.10 and 3.4.11 we have.

3.4.12 Proposition (EXACTNESS). The $\mathcal{J}$-singular cohomology theory satisfies the Exactness Axiom.

3.5 Induced Morphisms for Singular $\mathcal{J}$-Homology

3.5.1 Let $f:(X,A) \longrightarrow (Y,B)$ be a morphism of saturated $\mathcal{J}$-pairs. We shall give a procedure for assigning to f a pair $(\hat{f}_1, \hat{f}_2)$ of chain maps with the following properties:

(1) The following square commutes:

$$\begin{array}{ccc}
\sum\limits_{a,b} S_\#\big((Y_b,B_b)\underline{x}\mathcal{J}(a,b)\big)\otimes M_a & \xrightarrow{\Phi^{Y,B}} & \sum\limits_{c} S_\#(Y_c,B_c)\otimes M_c \\
\uparrow{\scriptstyle \hat{f}_1} & & \uparrow{\scriptstyle \hat{f}_2} \\
\sum\limits_{a,b} S_\#\big((X_b,A_b)\underline{x}\mathcal{J}(a,b)\big)\otimes M_a & \xrightarrow[\Phi^{X,A}]{} & \sum\limits_{c} S_\#(X_c,A_c)\otimes M_c
\end{array}$$

(2) If f is the identity morphism, then both $\hat{f}_1$ and $\hat{f}_2$ are the identity chain maps.

(3) If $f:(X,A) \longrightarrow (Y,B)$ and $g:(Y,B) \longrightarrow (Z,C)$ are morphisms of saturated $\mathcal{J}$-pairs, then

$$(g\circ f)^\wedge_1 = \hat{g}_1\circ\hat{f}_1, \text{ and } (g\circ f)^\wedge_2 = \hat{g}_2\circ\hat{f}_2.$$

We shall derive explicit formulas on $\hat{f}_1$ and $\hat{f}_2$, from which we immediately conclude properties (1)–(3). These formulas are essentially the same as have already been developed on in the course of obtaining the short exact sequence

$$0 \longrightarrow C^{\mathcal{J}}_{*}(A;\, \mathcal{M}) \xrightarrow{i_{\#}} C^{\mathcal{J}}_{*}(X;\, \mathcal{M}) \xrightarrow{j_{\#}} C^{\mathcal{J}}_{*}(X,A;\, \mathcal{M}) \longrightarrow 0$$

from the inclusions $i{:}(A,\varnothing) \longrightarrow (X,\varnothing)$ and $j{:}(X,\varnothing) \longrightarrow (X,A)$.

3.5.2 For $f{:}(X,A) \longrightarrow (Y,B)$ we use the following notation concerning morphisms induced on $(A,\varnothing)$ and $(X,\varnothing)$: $f^A = f|A{:}(A,\varnothing) \longrightarrow (Y,\varnothing)$, $f^X = f|X{:}(X,\varnothing) \longrightarrow (Y,\varnothing)$. For the remainder of this section let $f{:}(X,A) \longrightarrow (Y,B)$ be a fixed morphism of saturated $\mathcal{J}$–pairs and $\mathcal{M}{:}\mathcal{J} \longrightarrow \mathbf{Compl}_R$ a fixed covariant coefficient system on $\mathcal{J}$.

In the following lemma we use the notation from 3.3.2.

3.5.3 Lemma. Let $a,b \in \operatorname{ob} \mathcal{J}$. The following diagram commutates in ${}_{\mathbb{Z}}\mathbf{Compl}$.

$$\begin{tikzcd}
\mathcal{S}_{\#}\big((Y_b,B_b)\underline{\times}\mathcal{J}(a,b)\big) \arrow[r, "(\tilde{Y}_{ab},\tilde{B}_{ab})_{\#}"] & \mathcal{S}_{\#}(Y_a,B_a) \\
\mathcal{S}_{\#}\big((X_b,A_b)\underline{\times}\mathcal{J}(a,b)\big) \arrow[r, "(\tilde{X}_{ab},\tilde{A}_{ab})_{\#}"'] \arrow[u, "(f_b\underline{\times}1)_{\#}"] & \mathcal{S}_{\#}(X_a,A_a) \arrow[u, "f_{b\#}"']
\end{tikzcd}$$

Proof. By the naturality of the morphism $f{:}(X,A) \longrightarrow (Y,B)$ the diagram below commutes

$$\begin{tikzcd}
(Y_b,B_b)\underline{\times}\mathcal{J}(a,b) \arrow[r, "(\tilde{Y}_{ab},\tilde{B}_{ab})"] & (Y_a,B_a) \\
(X_b,A_b)\underline{\times}\mathcal{J}(a,b) \arrow[r, "(\tilde{X}_{ab},\tilde{A}_{ab})"'] \arrow[u, "f_b\underline{\times}1"] & (X_a,A_a) \arrow[u, "f_a"']
\end{tikzcd}$$

An application of the singular functor S to this diagram yields the commutativity of the diagram.

3.5.4 Lemma. Let $a,b \in \text{ob}\, J$. The following diagram is commutative in $\mathbf{Compl}_R$.

$$\begin{tikzcd}
S_\#\big((Y_b, B_b)\underline{x}J(a,b)\big) \otimes_{\mathbb{Z}} \mathbb{M}_a \arrow[r, "\alpha^{Y,B}_{ab}"] & S_\#(Y_a, B_a) \otimes_{\mathbb{Z}} \mathbb{M}_a \\
S_\#\big((X_b, A_b)\underline{x}J(a,b)\big) \otimes_{\mathbb{Z}} \mathbb{M}_a \arrow[r, "\alpha^{X,A}_{ab}"'] \arrow[u, "(f_b\underline{x}1)_\#\otimes 1"] & S_\#(X_a, A_a) \otimes_{\mathbb{Z}} \mathbb{M}_a \arrow[u, "f_{a\#}\otimes 1"']
\end{tikzcd}$$

Proof. A tensor product the diagram in 3.5.3 with $\mathbb{M}_a$ and the definition of α's in 3.3.4 yields the assertion.

3.5.5 Lemma. Let $a,b \in \text{ob}\, J$. The following diagram is commutative in $\mathbf{Compl}_R$.

$$\begin{tikzcd}
S_\#\big((Y_b, B_b)\underline{x}J(a,b)\big) \otimes_{\mathbb{Z}} \mathbb{M}_a \arrow[r, "\beta^{Y,B}_{ab}"] \arrow[d, "(f_b\underline{x}1)_\#\otimes 1"'] & S_\#(Y_b, B_b) \otimes_{\mathbb{Z}} \mathbb{M}_b \arrow[d, "f_{b\#}\otimes 1"] \\
S_\#\big((X_b, A_b)\underline{x}J(a,b)\big) \otimes_{\mathbb{Z}} \mathbb{M}_a \arrow[r, "\beta^{X,A}_{ab}"'] & S_\#(X_b, A_b) \otimes_{\mathbb{Z}} \mathbb{M}_b
\end{tikzcd}$$

Proof. By the direct calculation we have

$f_{b\#}\otimes 1_{\mathbb{M}_b}\left(\beta^{X,B}_{ab}([\sigma,\eta]\otimes m)\right) = f_{b\#}\otimes 1_{\mathbb{M}_b}\big([\sigma]\otimes\check{\mathbb{M}}(\eta)(m)\big) = [f_{b\#}(\sigma)]\otimes\check{\mathbb{M}}(\eta)(m)$ and

$$\beta^{X,B}_{ab}\Big((f_b\underline{x}1_{J(a,b)})_{\#}\otimes 1_{M_a}([\sigma,\eta]\otimes m)\Big) = \beta^{X,B}_{ab}([f_{b\#}(\sigma),\eta]\otimes m) = [f_{b\#}(\sigma)]\otimes\check{M}(\eta)(m).$$

3.5.6 Definition. Define

$$\hat{f}_1 : \sum_{a,b} S_{\#}\big((X_b,A_b)\underline{x}J(a,b)\big) \otimes_Z M_a \longrightarrow \sum_{a,b} S_{\#}\big((Y_b,B_b)\underline{x}J(a,b)\big) \otimes_Z M_a$$

by $\hat{f}_1 = \Big((f_b\underline{x}1_{J(a,b)})_{\#}\otimes 1_{M_a}\Big)_{a,b \in \text{ob } J}$.

Define

$$\hat{f}_2 : \sum_{c} S_{\#}(X_c,A_c) \otimes_Z M_c \longrightarrow \sum_{c} S_{\#}(Y_c,B_c) \otimes_Z M_c$$

by $\hat{f}_2 = \Big(f_{c\#}\otimes 1_{M_c}\Big)_{c \in \text{ob } J}$.

3.5.7 Proposition.

(1) The following square commutes in $\mathbf{Compl}_R$.

$$\begin{tikzcd}
\sum_{a,b} S_{\#}\big((Y_b,B_b)\underline{x}J(a,b)\big)\otimes M_a \arrow[r, "\Phi^{Y,B}"] & \sum_{c} S_{\#}(Y_c,B_c)\otimes M_c \\
\sum_{a,b} S_{\#}\big((X_b,A_b)\underline{x}J(a,b)\big)\otimes M_a \arrow[u, "\hat{f}_1"] \arrow[r, "\Phi^{X,A}"'] & \sum_{c} S_{\#}(X_c,A_c)\otimes M_c \arrow[u, "\hat{f}_2"']
\end{tikzcd}$$

(2) If f is an identity morphism, then both $\hat{f}_1$ and $\hat{f}_2$ are the identity chain maps.

(3) If $f:(X,A) \longrightarrow (Y,B)$ and $g:(Y,B) \longrightarrow (Z,C)$ are morphisms of saturated J-pairs, then

$$(g\circ f)^{\wedge}_1 = \hat{g}_1\circ\hat{f}_1, \text{ and } (g\circ f)^{\wedge}_2 = \hat{g}_2\circ\hat{f}_2.$$

Proof. (1) follows by combining commutative diagrams in 3.5.4 and 3.5.5; vertical arrows combine into $\hat{f}_1$ and $\hat{f}_2$ (see definitions in 3.5.6), and horizontal arrows combine into $\Phi^{X,A}$ and $\Phi^{Y,B}$. (2) and (3) are an immediate consequence of definitions in 3.5.6.

3.5.8 Definition. Define $f_\#:C^J_*(X,A;\ \mathbb{M}) \longrightarrow C^J_*(Y,B;\ \mathbb{M})$ as a chain map induced by the commutative diagram in 3.5.7(1) between $\mathrm{coker}(\Phi^{X,A}) = C^J_*(X,A;\ \mathbb{M})$ and $\mathrm{coker}(\Phi^{Y,B}) = C^J_*(Y,B;\ \mathbb{M})$.

3.5.9 Proposition. Let $f:(X,A) \longrightarrow (Y,B)$ be a morphism of saturated J-pairs and $\mathbb{M}:J \longrightarrow \mathbf{Compl}_R$ a covariant coefficient system on J. Then

(1) If f is an identity morphism, then $f_\#$ is the identity chain map.

(2) If $f:(X,A) \longrightarrow (Y,B)$ and $g:(Y,B) \longrightarrow (Z,C)$ are morphisms of saturated J-pairs, then $(g \circ f)_\# = g_\# \circ f_\#$.

Proof. The claims follow directly from 3.5.7 and from the properties of the cokernel sequence.

3.5.10 Definition. Define $f_*:H^J_*(X,A;\ \mathbb{M}) \longrightarrow H^J_*(Y,B;\ \mathbb{M})$ by applying the homology functor to the chain map $f_\#:C^J_*(X,A;\ \mathbb{M}) \longrightarrow C^J_*(Y,B;\ \mathbb{M})$.

3.5.11 Proposition. Let $f:(X,A) \longrightarrow (Y,B)$ be a morphism of saturated J-pairs and $\mathbb{M}:J \longrightarrow \mathbf{Compl}_R$ a covariant coefficient system on J. Then

(1) If f is an identity morphism, then f_* is the identity of a graded module.

(3) If $f:(X,A) \longrightarrow (Y,B)$ and $g:(Y,B) \longrightarrow (Z,C)$ are morphisms of saturated

$\mathcal{J}$–pairs, then $(g\circ f)_* = g_*\circ f_*$.

Proof. The assertions follow directly from 3.5.9 and from the properties of the homology functor of chain complexes.

3.5.12 Corollary. Let $f:(X,A) \longrightarrow (Y,B)$ be a morphism of saturated $\mathcal{J}$–pairs and $\mathbb{M}:\mathcal{J} \longrightarrow \mathbf{Compl}_R$ a covariant coefficient system on $\mathcal{J}$. Then the following diagram commutes in $\mathbf{Compl}_R$.

$$\begin{tikzcd}
0 \arrow[r] & C_*^{\mathcal{J}}(B;\mathbb{M}) \arrow[r, "i_{(Y,B)\#}"] & C_*^{\mathcal{J}}(Y;\mathbb{M}) \arrow[r, "j_{(Y,B)\#}"] & C_*^{\mathcal{J}}(Y,B;\mathbb{M}) \arrow[r] & 0 \\
0 \arrow[r] & C_*^{\mathcal{J}}(A;\mathbb{M}) \arrow[r, "i_{(X,A)\#}"] \arrow[u, "f^A_\#"] & C_*^{\mathcal{J}}(X;\mathbb{M}) \arrow[r, "j_{(X,A)\#}"] \arrow[u, "f^X_\#"'] & C_*^{\mathcal{J}}(X,A;\mathbb{M}) \arrow[r] \arrow[u, "f_\#"'] & 0
\end{tikzcd}$$

Proof. Pairs (X,A) and (Y,B) induce inclusions $i_{(X,A)}:(A,\emptyset) \longrightarrow (X,\emptyset)$, $i_{(Y,B)}:(B,\emptyset) \longrightarrow (Y,\emptyset)$, $j_{(X,A)}:(X,\emptyset) \longrightarrow (X,A)$, and $j_{(Y,B)}:(Y,\emptyset) \longrightarrow (Y,B)$. Note that in 3.3.12 we already defined $i_{(X,A)\#}$, $i_{(Y,B)\#}$, $j_{(X,A)\#}$, and $j_{(Y,B)\#}$ using in fact definitions from 3.5.8. The following diagram of saturated $\mathcal{J}$–pairs

$$\begin{tikzcd}
(B,\emptyset) \arrow[r, "i_{(Y,B)}"] & (Y,\emptyset) \arrow[r, "j_{(Y,B)}"] & (Y,B) \\
(A,\emptyset) \arrow[r, "i_{(X,A)}"'] \arrow[u, "f^A"] & (X,\emptyset) \arrow[r, "j_{(X,A)}"'] \arrow[u, "f^X"'] & (X,A) \arrow[u, "f"']
\end{tikzcd}$$

commutes, and 3.5.9 and 3.3.13 yield the corollary.

3.6 Induced Morphisms for Singular $\mathcal{J}$–Cohomology

3.6.1 Let $f:(X,A) \longrightarrow (Y,B)$ be a morphism of saturated $\mathcal{J}$–pairs. Let $\kappa:\mathcal{J}^{op} \longrightarrow {}_R\mathbf{CoCompl}$ be a fixed contravariant coefficient system on $\mathcal{J}$. We shall give a procedure to assign to f a pair $(\check{f}_1, \check{f}_2)$ of cochain maps with the following properties.

3.6.2 Proposition.

(1) The following square commutes in ${}_R\mathbf{CoCompl}$:

$$\begin{array}{ccc}
\prod_a \underline{\mathrm{Hom}}(S_\#(X_a,A_a), \kappa_a) & \xrightarrow{\psi^{X,A}} & \prod_{b,c} \underline{\mathrm{Hom}}(S_\#((X_c,A_c)\underline{x}\mathcal{J}(b,c)), \kappa_b) \\
\uparrow{\scriptstyle \check{f}_1} & & \uparrow{\scriptstyle \check{f}_2} \\
\prod_a \underline{\mathrm{Hom}}(S_\#(Y_a,B_a), \kappa_a) & \xrightarrow[\psi^{Y,B}]{} & \prod_{b,c} \underline{\mathrm{Hom}}(S_\#((Y_c,B_c)\underline{x}\mathcal{J}(b,c)), \kappa_b)
\end{array}$$

(2) If f is the identity morphism, then both $\check{f}_1$ and $\check{f}_2$ are the identity chain maps.

(3) If $f:(X,A) \longrightarrow (Y,B)$ and $g:(Y,B) \longrightarrow (Z,C)$ are morphisms of saturated $\mathcal{J}$–pairs, then

$$(g\circ f)\check{}_1 = \check{g}_1\circ\check{f}_1, \text{ and } (g\circ f)\check{}_2 = \check{g}_2\circ\check{f}_2.$$

3.6.3 The proof is given in 3.6.7. We shall derive explicit formulas for $\check{f}_1$ and $\check{f}_2$, from which we immediately conclude properties (1)–(3). These formulas are essentially the same as we already developed on a way of obtaining the short exact sequence

$$0 \longrightarrow C^*_J(X,A;\ \mathbb{M}) \xrightarrow{j^\#} C^*_J(X;\ \mathbb{M}) \xrightarrow{i^\#} C^*_J(A;\ \mathbb{M}) \longrightarrow 0$$

from the inclusions $i:(A,\varnothing) \longrightarrow (X,\varnothing)$ and $j:(X,\varnothing) \longrightarrow (X,A)$.

3.6.4 Lemma. Let $a,b \in \text{ob } J$. The following diagram is commutative in ${}_R$**CoCompl.**

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(S_\#(X_a,A_a),\ \mathbb{K}_a) \arrow[r, "\phi^{X,A}_{ab}"] & \underline{\mathrm{Hom}}(S_\#((X_b,A_b)\underline{x}J(a,b)),\ \mathbb{K}_a) \\
\underline{\mathrm{Hom}}(S_\#(Y_a,B_a),\ \mathbb{K}_a) \arrow[r, "\phi^{Y,B}_{ab}"'] \arrow[u, "\underline{\mathrm{Hom}}(f_{a\#},\ 1_{\mathbb{K}_a})"] & \underline{\mathrm{Hom}}(S_\#((Y_b,B_b)\underline{x}J(a,b)),\ \mathbb{K}_a) \arrow[u, "\underline{\mathrm{Hom}}((f_b\underline{x}1)_\#,1_{\mathbb{K}_a})"']
\end{tikzcd}$$

Proof. The proof is done by taking $\underline{\mathrm{Hom}}$ of the diagram in 3.5.3 with $\mathbb{K}_a$ and the assertion follows from the definition of ϕ's in 3.4.4.

3.6.5 Lemma. The following diagram is commutative in ${}_R$**CoCompl.**

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(S_\#(X_b,A_b),\ \mathbb{K}_b) \arrow[r, "\psi^{X,A}_{ab}"] & \underline{\mathrm{Hom}}(S_\#((X_b,A_b)\underline{x}J(a,b)),\ \mathbb{K}_a) \\
\underline{\mathrm{Hom}}(S_\#(Y_b,B_b),\ \mathbb{K}_b) \arrow[r, "\psi^{Y,B}_{ab}"'] \arrow[u, "\underline{\mathrm{Hom}}(f_{b\#},\ 1_{\mathbb{K}_b})"] & \underline{\mathrm{Hom}}(S_\#((Y_b,B_b)\underline{x}J(a,b)),\ \mathbb{K}_a) \arrow[u, "\underline{\mathrm{Hom}}((f_b\underline{x}1)_\#,1_{\mathbb{K}_a})"']
\end{tikzcd}$$

Proof. Let $\bar{g} = \{g_{p,q}: \mathcal{S}_p(Y_b,B_b) \longrightarrow \mathbb{K}_{b_q}\} \in \underline{\mathrm{Hom}}(\mathcal{S}_p(Y_b,B_b), \mathbb{K}_b)$ and $[(\sigma,\eta)] \in \mathcal{S}_p(X_b,A_b)$. See in 3.4.4 the definition of ψ's.

Then $\left[\psi^{X,B}_{ab}\ (\underline{\mathrm{Hom}}(f_{b\#},1_{\mathbb{K}_b})(\bar{g}))\right]_{p,q}([(\sigma,\eta)]) =$

$\left[\psi^{X,B}_{ab}\ (\{g_{i,j}\circ f_{a\#}:\mathcal{S}_i(X_b,A_b) \longrightarrow \mathbb{K}_{b_j}\})\right]_{p,q}([(\sigma,\eta)]) = \check{\mathbb{K}}(\eta)(g_{p,q}\circ f_{a\#}(\sigma))$ and

$\left[\underline{\mathrm{Hom}}((f_b\underline{x}1_{J(a,b)})_{\#},1_{\mathbb{K}_a})\ (\psi^{Y,B}_{ab}(\bar{g}))\right]_{p,q}([(\sigma,\eta)]) =$

$\left[\{\psi^{Y,B}_{ab}(\bar{g})_{i,j}\circ(f_b\underline{x}1_{J(a,b)})_{\#}:\mathcal{S}_i(X_b,A_b) \longrightarrow \mathbb{K}_{a_j}\}\right]_{p,q}([(\sigma,\eta)]) =$

$\psi^{Y,B}_{ab}(\bar{g})_{p,q}\circ(f_b\underline{x}1_{J(a,b)})_{\#}([(\sigma,\eta)]) = \psi^{Y,B}_{ab}(\bar{g})_{p,q}(f_{b\#}(\sigma),\eta) =$

$\check{\mathbb{K}}(\eta)(g_{p,q}(f_{a\#}(\sigma)))$ which proves the assertion.

3.6.6 Definition. Define

$$\check{f}_1: \prod_a \underline{\mathrm{Hom}}(\mathcal{S}_{\#}(Y_a,B_a), \mathbb{K}_a) \longrightarrow \prod_a \underline{\mathrm{Hom}}(\mathcal{S}_{\#}(X_a,A_a), \mathbb{K}_a)$$

by $\check{f}_1 = \left(\underline{\mathrm{Hom}}(f_{a\#}, 1_{\mathbb{K}_a})\right)_{a \in \mathrm{ob}\, J}$.

Define

$$\check{f}_2: \prod_{b,c} \underline{\mathrm{Hom}}(\mathcal{S}_{\#}((Y_c,B_c)\underline{x}J(b,c)), \mathbb{K}_b) \longrightarrow \prod_{b,c} \underline{\mathrm{Hom}}(\mathcal{S}_{\#}((X_c,A_c)\underline{x}J(b,c)), \mathbb{K}_b)$$

by $\check{f}_2 = \left(\underline{\mathrm{Hom}}((f_c\underline{x}1_{J(b,c)})_{\#}, 1_{\mathbb{K}_a})\right)_{b,c \in \mathrm{ob}\, J}$.

3.6.7 Proof of Proposition 3.6.2. (1) follows by combining the commutative diagrams in 3.6.4 and 3.6.5; vertical arrows combine into $\check{f}_1$ and $\check{f}_2$ (see definitions in 3.6.6), and horizontal arrows combine into $\Psi^{X,A}$ and $\Psi^{Y,B}$. (2)

and (3) are an immediate consequence of definitions in 3.6.6.

3.6.8 Definition. Define $f^{\#}:C^{\#}_{J}(Y,B;\ \mathbb{K}) \longrightarrow C^{\#}_{J}(X,A;\ \mathbb{K})$ as a cochain map induced by the commutative diagram in 3.6.2(1) between kernels $\ker(\Psi^{Y,B}) = C^{\#}_{J}(Y,B;\ \mathbb{K})$ and $\ker(\Psi^{X,A}) = C^{\#}_{J}(X,A;\ \mathbb{K})$.

3.6.9 Proposition. Let $f:(X,A) \longrightarrow (Y,B)$ be a morphism of saturated J-pairs and $\mathbb{K}:J^{op} \longrightarrow_{R}$**CoCompl** a contravariant coefficient system on J. Then

(1) If f is the identity morphism, then $f^{\#}$ is the identity chain map.

(2) If $f:(X,A) \longrightarrow (Y,B)$ and $g:(Y,B) \longrightarrow (Z,C)$ are morphisms of saturated J-pairs, then $(g \circ f)^{\#} = f^{\#} \circ g^{\#}$.

Proof. The assertions follow directly from 3.6.2 and properties of kernels.

3.6.10 Definition. Define $f_*:H^{*}_{J}(Y,B;\ \mathbb{K}) \longrightarrow H^{*}_{J}(X,A;\ \mathbb{K})$ by applying the homology functor to the cochain map $f^{\#}:C^{\#}_{J}(Y,B;\ \mathbb{K}) \longrightarrow C^{\#}_{J}(X,A;\ \mathbb{K})$.

3.6.11 Theorem. Let $f:(X,A) \longrightarrow (Y,B)$ be a morphism of saturated J-pairs and $\mathbb{K}:J^{op} \longrightarrow_{R}$**CoCompl** a contravariant coefficient system on J. Then

(1) If f is the identity morphism, then f^{*} is the identity on a graded module.

(2) If $f:(X,A) \longrightarrow (Y,B)$ and $g:(Y,B) \longrightarrow (Z,C)$ are morphisms of saturated J-pairs, then $(g \circ f)^{*} = f^{*} \circ g^{*}$.

Proof. The assertions follow directly from 3.6.9 and properties of the homology functor of cochain complexes.

3.6.12 Corollary. Let $f:(X,A) \longrightarrow (Y,B)$ be a morphism of saturated $\mathcal{J}$-pairs and $\mathbb{K}:\mathcal{J}^{op} \longrightarrow {}_R\mathbf{CoCompl}$ a contravariant coefficient system on $\mathcal{J}$. Then the following diagram commutes in ${}_R\mathbf{CoCompl}$.

$$\begin{tikzcd}
0 \arrow[r] & C^*_{\mathcal{J}}(X,A;\mathbb{M}) \arrow[r, "j_{(X,A)}^{\#}"] & C^*_{\mathcal{J}}(X;\mathbb{M}) \arrow[r, "i_{(X,A)}^{\#}"] & C^*_{\mathcal{J}}(A;\mathbb{M}) \arrow[r] & 0 \\
0 \arrow[r] & C^*_{\mathcal{J}}(Y,B;\mathbb{M}) \arrow[r, "j_{(Y,B)}^{\#}"] \arrow[u, "f^{\#}"] & C^*_{\mathcal{J}}(Y;\mathbb{M}) \arrow[r, "i_{(Y,B)}^{\#}"] \arrow[u, "f^{X\#}"'] & C^*_{\mathcal{J}}(B;\mathbb{M}) \arrow[r] \arrow[u, "f^{A\#}"'] & 0
\end{tikzcd}$$

Proof. The pairs (X,A) and (Y,B) induce inclusions $i_{(X,A)}:(A,\varnothing) \longrightarrow (X,\varnothing)$, $i_{(Y,B)}:(B,\varnothing) \longrightarrow (Y,\varnothing)$, $j_{(X,A)}:(X,\varnothing) \longrightarrow (X,A)$, and $j_{(Y,B)}:(Y,\varnothing) \longrightarrow (Y,B)$. Note that we $i_{(X,A)}^{\#}$, $i_{(Y,B)}^{\#}$, $j_{(X,A)}^{\#}$, and $j_{(Y,B)}^{\#}$ already defined in 3.4.10 using definition from 3.6.8. The following diagram of saturated $\mathcal{J}$-pairs commutes

$$\begin{tikzcd}
(B,\varnothing) \arrow[r, "i_{(Y,B)}"] & (Y,\varnothing) \arrow[r, "j_{(Y,B)}"] & (Y,B) \\
(A,\varnothing) \arrow[r, "i_{(X,A)}"'] \arrow[u, "f^{A}"] & (X,\varnothing) \arrow[r, "j_{(X,A)}"'] \arrow[u, "f^{X}"'] & (X,A) \arrow[u, "f"']
\end{tikzcd}$$

and thus 3.6.9 and 3.4.10 gives us the conclusion.

3.7 Homotopy Axiom for Singular $\mathcal{J}$-Homology

In this section we shall prove singular $\mathcal{J}$-homology satisfies the homotopy axiom.

3.7.1 First we sketch the construction of a chain homotopy induced by a given homotopy between continuous maps in **Top**. Let $f,g:X \longrightarrow Y$ be two continuous maps of topological spaces X and Y. Assume that there is a homotopy $F:f_0 \approx f_1$, $F:I \times X \longrightarrow Y$ ($I = [0,1]$). Recall that the standard 1-complex $\bar{\Delta}(1)$ can be obtained as the free chain $\mathbb{Z}$-complex generated by two (singular) 0-simplexes $\{0\}$ and $\{1\}$, in dimension zero, and by Ω, the 1-dimensional simplex $[0,1]$, in dimension one. Its differential is taken to be zero in all dimensions but one, where $\partial\Omega = \{0\} - \{1\}$. From the above interpretation of $\bar{\Delta}(1)$, it is immediate that it is a subcomplex of $S_{\#}(I)$. The canonical inclusions $\hat{i}_0,\hat{i}_1:S_{\#}(X) \longrightarrow \bar{\Delta}(1)\otimes S_{\#}(X)$ are defined by $\sigma \longrightarrow \{0\}\otimes\sigma$ and $\sigma \longrightarrow \{1\}\otimes\sigma$, respectively. Given F we shall construct a chain map $\hat{F}:\bar{\Delta}(1)\otimes S_{\#}(X) \longrightarrow S_{\#}(Y)$, such that $\hat{F}\circ\hat{i}_0 = f_{0\#}$, $\hat{F}\circ\hat{i}_1 = f_{1\#}$, and note that the existence of $\hat{F}$ is equivalent to the existence of a chain homotopy between $f_{0\#}$ and $f_{1\#}$ (see **[Baues]** p. 41).

Observe that $\bar{\Delta}(1)\otimes S_{\#}(X)$ is a chain subcomplex of $S_{\#}(I)\otimes S_{\#}(X)$. Let ε be the Eilenberg-Zilber map, a chain homotopy equivalence from $S_{\#}(I)\otimes S_{\#}(X)$ to $S_{\#}(I\underline{x}X)$ (see **[MacLane 1]** p.239). For a topological space X define $\hat{\varepsilon} = \hat{\varepsilon}_X:\bar{\Delta}(1)\otimes S_{\#}(X) \longrightarrow S_{\#}(I\underline{x}X)$ by $\hat{\varepsilon} = \varepsilon|\bar{\Delta}(1)\otimes S_{\#}(X)$. Explicitly, $\hat{\varepsilon}$ is given on generators by

$$\hat{\varepsilon}(\{0\}\otimes \sigma) = (\{0\}, \sigma), \qquad \hat{\varepsilon}(\{1\}\otimes \sigma) = (\{1\}, \sigma)$$

where on the right side both $\{0\}$ and $\{1\}$ are treated as singular $|\sigma|$-simplexes in $I = [0,1]$ concentrated at points 0 and 1, respectively, and

$$\hat{\varepsilon}(\Omega \otimes \sigma) = \sum_{k=0}^{n}(-1)^k(s_n \ldots s_{k+1}s_{k-1} \ldots s_0\Omega,\ s_k\sigma),$$

where s_k's are the degeneracy operators of the singular simplicial structures on I and X (see [May 2], p. 133). By checking directly we obtain the following lemma.

3.7.2 Lemma.

(1) The canonical inclusions $\hat{i}_0, \hat{i}_1 : S_\#(X) \longrightarrow \bar{\Delta}(1) \otimes S_\#(X)$ are related to the top and bottom inclusions $i_0, i_1 : X \longrightarrow I\underline{x}X$, $x \longrightarrow (0,x)$, $x \longrightarrow (1,x)$, by

$$\hat{\varepsilon} \circ \hat{i}_0 = i_{0\#}, \quad \hat{\varepsilon} \circ \hat{i}_1 = i_{1\#}.$$

(2) The chain $\hat{\varepsilon}_X$ is natural in X, that is, if $h:X \longrightarrow Z$ is a continuous function between topological spaces X and Z, then the following diagram commutes:

$$\begin{array}{ccc} \bar{\Delta}(1) \otimes S_\#(Z) & \xrightarrow{\hat{\varepsilon}_Z} & S_\#(I\underline{x}Z) \\ {\scriptstyle 1 \otimes h_\#} \uparrow & & \uparrow {\scriptstyle (1\underline{x}h)_\#} \\ \bar{\Delta}(1) \otimes S_\#(X) & \xrightarrow[\hat{\varepsilon}_X]{} & S_\#(I\underline{x}X). \end{array}$$

3.7.3 Definition. Define $\hat{F} : \bar{\Delta}(1) \otimes S_\#(X) \longrightarrow S_\#(X)$ by $\hat{F} = F_\# \circ \hat{\varepsilon}$. Note that $\hat{F} \circ \hat{i}_k = F_\# \circ \hat{\varepsilon} \circ \hat{i}_k = F_\# \circ i_{k\#} = f_{k\#}$, $k = 0,1$, and therefore $\hat{F}$ is a homotopy between $f_{0\#}$ and $f_{1\#}$.

3.7.4 Let $f^0, f^1 : (X,A) \longrightarrow (Y,B)$ be two homotopic morphisms of saturated

$\mathcal{J}$-pairs with a homotopy $G:(IX,IA) \longrightarrow (Y,B)$, $G:f^0 \simeq f^1$. Recall that f^0 gives rise to a pair of chain maps $(\hat{f}^0_1, \hat{f}^0_2)$ such that the left square of the following diagram

$$\begin{tikzcd}
\sum\limits_{a,b} S_\#((Y_b,B_b)\underline{x}\mathcal{J}(a,b))\otimes M_a \arrow[r, "\Phi^{Y,B}"] & \sum\limits_{c} S_\#(Y_c,B_c)\otimes M_c \arrow[r] & C^{\mathcal{J}}_*(Y,B;\, M) \\
\sum\limits_{a,b} S_\#((X_b,A_b)\underline{x}\mathcal{J}(a,b))\otimes M_a \arrow[r, "\Phi^{X,A}"'] \arrow[u, "\hat{f}^0_1"] & \sum\limits_{c} S_\#(X_c,A_c)\otimes M_c \arrow[r] \arrow[u, "\hat{f}^0_2"'] & C^{\mathcal{J}}_*(X,A;\, M) \arrow[u, "f^0_\#"']
\end{tikzcd}$$

commutes, and the chain map $f^0_\#:C^{\mathcal{J}}_*(X,A;\, M) \longrightarrow C^{\mathcal{J}}_*(Y,B;\, M)$, a map between cokernels $\operatorname{coker}(\Phi^{X,A}) = C^{\mathcal{J}}_*(X,A;\, M)$ and $\operatorname{coker}(\Phi^{Y,B}) = C^{\mathcal{J}}_*(Y,B;\, M)$, is induced by $(\hat{f}^0_1, \hat{f}^0_2)$. Similarly, f^1 gives rise to a pair $(\hat{f}^1_1, \hat{f}^1_2)$ and the induced chain map $f^1_\#:C^{\mathcal{J}}_*(X,A;\, M) \longrightarrow C^{\mathcal{J}}_*(Y,B;\, M)$ with the same properties as just described in the case of f^0 (see Proposition 3.5.2). We shall prove the following proposition.

3.7.5 Proposition. Let $f^0,f^1:(X,A) \longrightarrow (Y,B)$ be two homotopic morphisms of $\mathcal{J}$-pairs. Then there exists a pair $(\hat{G}_1,\hat{G}_2)$ of homotopies, $\hat{G}_1:\hat{f}^0_1 \simeq \hat{f}^1_1$, $\hat{G}_2:\hat{f}^0_2 \simeq \hat{f}^1_2$, such that, for $k = 0,1$, the following diagram commutes:

$$\begin{tikzcd}
\sum\limits_{a,b} S_\#\big((Y_b, B_b)\underline{x}\mathcal{J}(a,b)\big)\otimes\mathbb{M}_a \arrow[r, "\Phi^{Y,A}"] & \sum\limits_{c} S_\#(Y_c,B_c)\otimes\mathbb{M}_c \\
\bar{\Delta}(1)\otimes\Big[\sum\limits_{a,b} S_\#\big((X_b,A_b)\underline{x}\mathcal{J}(a,b)\big)\otimes\mathbb{M}_a\Big] \arrow[u, "\hat{G}_1"] \arrow[r, "1\otimes\Phi^{X,A}"'] & \bar{\Delta}(1)\otimes\Big[\sum\limits_{c} S_\#(X_c,A_c)\otimes\mathbb{M}_c\Big] \arrow[u, "\hat{G}_2"'] \\
\sum\limits_{a,b} S_\#\big((X_b, A_b)\underline{x}\mathcal{J}(a,b)\big)\otimes\mathbb{M}_a \arrow[u, "\hat{i}_k"] \arrow[r, "\Phi^{X,A}"'] \arrow[uu, bend left=70, "\hat{f}^k_1"] & \sum\limits_{c} S_\#(X_c,A_c)\otimes\mathbb{M}_c \arrow[u, "\hat{i}_k"'] \arrow[uu, bend right=70, "\hat{f}^k_2"']
\end{tikzcd} .$$

Proof. The proof is given in 3.7.14.

3.7.6 Proposition. If $f^0,f^1:(X,A) \longrightarrow (Y,B)$ are two homotopic morphisms of $\mathcal{J}$-pairs, then the induced chain maps $f^0_\#,f^1_\#:C^{\mathcal{J}}_*(X,A;\ \mathbb{M}) \longrightarrow C^{\mathcal{J}}_*(Y,B;\ \mathbb{M})$ are homotopic.

Proof. By the Proposition 3.7.5 there is a pair $(\hat{G}_1,\hat{G}_2)$ of homotopies. Then the commutativity of the upper middle square in the diagram 3.7.5 induces the existence of a chain map $\hat{G}:\mathrm{coker}\big(1\otimes\Phi^{X,A}\big) \longrightarrow \mathrm{coker}\big(\Phi^{Y,B}\big)$. Note that $\bar{\Delta}(1)$ is a free chain complex and therefore a functor $\bar{\Delta}(1)\otimes_{\mathbf{Z}}\_$ is exact. Thus $\mathrm{coker}\big(1_{\bar{\Delta}(1)}\otimes_{\mathbf{Z}}\Phi^{X,A}\big) = \bar{\Delta}(1)\otimes_{\mathbf{Z}}\mathrm{coker}\big(\Phi^{X,A}\big)$ and it is easy to see that $\hat{G}$ is a homotopy between $f^0_\#$ and $f^1_\#$.

From Proposition 3.7.6 we have an immediate corollary.

3.7.7 Theorem (HOMOTOPY). If $f^0,f^1:(X,A) \longrightarrow (Y,B)$ are two homotopic morphisms of $\mathcal{J}$-pairs, then $f^0_* = f^1_*:H^{\mathcal{J}}_*(X,A;\ \mathbb{M}) \longrightarrow H^{\mathcal{J}}_*(Y,B;\ \mathbb{M})$.

3.7.8 Lemma. Let $f^0, f^1 : (X,A) \longrightarrow (Y,B)$ be morphisms of J-pairs such that there is a homotopy $G : f^0 \simeq f^1$, $G : (IX, IA) \longrightarrow (Y,B)$. Let $a,b \in \text{ob}\, J$. The following diagram commutes.

$$\begin{tikzcd}
S_\#\big((Y_b, B_b)\underline{x}J(a,b)\big) \arrow[r, "(\tilde{Y}_{ab}, \tilde{B}_{ab})_\#"] & S_\#(Y_a, B_a) \\
S_\#\big(I\underline{x}(X_b, A_b)\underline{x}J(a,b)\big) \arrow[u, "(G_b\underline{x}1)_\#"] \arrow[r, "(1\underline{x}(\tilde{X}_{ab}, \tilde{A}_{ab}))_\#"] & S_\#\big(I\underline{x}(X_a, A_a)\big) \arrow[u, "G_{a\#}"'] \\
\bar{\Delta}(1)\otimes S_\#\big((X_b, A_b)\underline{x}J(a,b)\big) \arrow[u, "\hat{\varepsilon}"] \arrow[r, "1\otimes(\tilde{X}_{ab}, \tilde{A}_{ab})_\#"'] & \bar{\Delta}(1)\otimes S_\#(X_a, A_a) \arrow[u, "\hat{\varepsilon}"']
\end{tikzcd}$$

Proof. If $a,b \in \text{ob}\, J$, the naturality of G is expressed as

$$G_a : I\underline{x}(X_a, A_a) \longrightarrow (Y_a, B_a) \qquad \begin{tikzcd}
(Y_b, B_b)\underline{x}J(a,b) \arrow[r, "(\tilde{Y}_{ab}, \tilde{B}_{ab})"] & (Y_a, B_a) \\
I\underline{x}(X_b, A_b)\underline{x}J(a,b) \arrow[u, "G_b\underline{x}1"] \arrow[r, "1\underline{x}(\tilde{X}_{ab}, \tilde{A}_{ab})"'] & I\underline{x}(X_a, A_a). \arrow[u, "G_a"']
\end{tikzcd}$$

which gives the commutativity of the upper square. The bottom square commutes by 3.7.2.

3.7.9 Note that the chain map $G_{b\#}\circ\hat{\varepsilon} : \bar{\Delta}(1)\otimes S_\#(X_b, A_b) \longrightarrow S_\#(Y_b, B_b)$ is a homotopy between $f^0_{b\#}$ and $f^1_{b\#}$, and the chain map

$$(G_{b\#}\underline{x}1)\circ\hat{\varepsilon} : \bar{\Delta}(1)\otimes S_\#\big((X_b, A_b)\underline{x}J(a,b)\big) \longrightarrow S_\#\big((Y_b, B_b)\underline{x}J(a,b)\big)$$

is a homotopy between $(f^0_{b}\underline{x}1_{J(a,b)})_\#$ and $(f^1_{b}\underline{x}1_{J(a,b)})_\#$. The following commutative diagram summarize relations between all these chain maps and $(\tilde{X}_{ab},\tilde{A}_{ab})_\#$ and $(\tilde{Y}_{ab},\tilde{B}_{ab})_\#$.

$$\begin{tikzcd}
S_\#\big((Y_b,B_b)\underline{x}J(a,b)\big) \arrow[r, "(\tilde{Y}_{ab},\tilde{B}_{ab})_\#"] & S_\#(Y_a,B_a) \\
\bar{A}(1)\otimes S_\#\big((X_b,A_b)\underline{x}J(a,b)\big) \arrow[u, "(G_b\underline{x}1)_\#\circ\hat{\varepsilon}"] \arrow[r, "1\otimes(\tilde{X}_{ab},\tilde{A}_{ab})_\#"'] & \bar{A}(1)\otimes S_\#(X_a,A_a) \arrow[u, "G_{a\#}\circ\hat{\varepsilon}"'] \\
S_\#\big((X_b,A_b)\underline{x}J(a,b)\big) \arrow[u, "\hat{i}_k"] \arrow[uu, bend left=60, "(f^k_b\underline{x}1)_\#"] \arrow[r, "(\tilde{X}_{ab},\tilde{A}_{ab})_\#"'] & S_\#(X_a,A_a) \arrow[u, "\hat{i}_k"'] \arrow[uu, bend right=60, "f^k_{a\#}"']
\end{tikzcd},$$

where $k = 0,1$.

Tensoring of the diagram in 3.3.7 with $\mathbb{M}_a$ gives us the following lemma.

3.7.10 Lemma. Let $a \in \operatorname{ob} J$.

(1) $G_{a\#}\circ\hat{\varepsilon}\otimes 1_{\mathbb{M}_a}$ is a homotopy between $f^1_{a\#}\otimes 1_{\mathbb{M}_a}$ and $f^2_{a\#}\otimes 1_{\mathbb{M}_a}$.

(2) $(G_{b\#}\underline{x}1_{J(a,b)})\circ\hat{\varepsilon}\otimes 1_{\mathbb{M}_a}$ is a homotopy between $(f^1_{b}\underline{x}1_{J(a,b)})_\#\otimes 1_{\mathbb{M}_a}$ and $(f^2_{b}\underline{x}1_{J(a,b)})_\#\otimes 1_{\mathbb{M}_a}$.

(3) The following diagram commutes

$$\begin{tikzcd}
S_{\#}\big((Y_b, B_b)\underline{x}J(a,b)\big)\otimes\mathbb{M}_a \arrow[r, "\alpha^{Y,B}_{ab}"] & S_{\#}(Y_a, B_a)\otimes\mathbb{M}_a \\
\bar{\Delta}(1)\otimes S_{\#}\big((X_b, A_b)\underline{x}J(a,b)\big)\otimes\mathbb{M}_a \arrow[u, "(G_b\underline{x}1)_{\#}\circ\hat{\varepsilon}\otimes 1"] \arrow[r, "1\otimes\alpha^{X,A}_{ab}"'] & \bar{\Delta}(1)\otimes S_{\#}(X_c, A_a)\otimes\mathbb{M}_a \arrow[u, "G_{a\#}\circ\hat{\varepsilon}\otimes 1"'] \\
S_{\#}\big((X_b, A_b)\underline{x}J(a,b)\big)\otimes\mathbb{M}_a \arrow[u, "\hat{i}_k"] \arrow[r, "\alpha^{X,A}_{ab}"'] \arrow[uu, bend left=60, "(f^k_b\underline{x}1)_{\#}\otimes 1"] & S_{\#}(X_a, A_a)\otimes\mathbb{M}_a \arrow[u, "\hat{i}_k"'] \arrow[uu, bend right=60, "f^k_{a\#}\otimes 1"']
\end{tikzcd}$$

where $k = 0,1$. Here and in the sequel the same notation $\hat{i}_k$, used in the form $\hat{i}_k:\mathbb{A} \longrightarrow \bar{\Delta}(1)\otimes\mathbb{A}$ in figures, is always understood to be one of two canonical inclusions of a shown chain complex $\mathbb{A}$ into $\bar{\Delta}(1)\otimes\mathbb{A}$ (see 3.7.1 for the definition of $\hat{i}_k$ in the case of $\mathbb{A} = S_{\#}(X)$).

Let $a,b \in \text{ob } J$. Define

$$\beta^{IX,IA}_{ab}:S_{\#}\big(I\underline{x}(X_b,A_b)\underline{x}J(a,b)\big) \otimes_{\mathbb{Z}} \mathbb{M}_b \longrightarrow S_{\#}\big(I\underline{x}(X_b,A_b)\big) \otimes_{\mathbb{Z}} \mathbb{M}_b,$$

by $\beta^{IX,IA}_{ab}\big([\sigma,\eta,\omega]\otimes m\big) = [\sigma,\eta]\otimes\check{\mathbb{M}}(\omega)(m)$. As in 3.3.7 we have that $\beta^{IX,IA}_{ab}$ is a chain map.

3.7.11 Lemma. The following diagram commutes.

$$\begin{tikzcd}
S_{\#}\big((Y_b, B_b)\underline{x}J(a,b)\big)\otimes \mathbb{M}_a \arrow[r, "\beta^{Y,B}_{ab}"] & S_{\#}(Y_b, B_b)\otimes \mathbb{M}_b \\
S_{\#}\big(I\,\underline{x}\,(X_b, A_b)\underline{x}J(a,b)\big)\otimes \mathbb{M}_a \arrow[r, "\beta^{IX,IA}_{ab}"] \arrow[u, "(G_b\underline{x}1)_{\#}\otimes 1"] & S_{\#}\big(I\underline{x}(X_b, A_b)\big)\otimes \mathbb{M}_b \arrow[u, "G_{b\#}\otimes 1"'] \\
\bar{\Delta}(1)\otimes S_{\#}\big((X_b, A_b)\underline{x}J(a,b)\big)\otimes \mathbb{M}_a \arrow[r, "1\otimes\beta^{X,A}_{ab}"'] \arrow[u, "\hat{\varepsilon}\otimes 1"] & \bar{\Delta}(1)\otimes S_{\#}(X_b, A_b)\otimes \mathbb{M}_b \arrow[u, "\hat{\varepsilon}\otimes 1"']
\end{tikzcd}$$

Proof. We show that both squares commute by checking directly on generators.

The upper square commutes since

$$\beta^{Y,B}_{ab}\Big((G_b\underline{x}1)_{\#}\otimes 1([\sigma,\eta,\omega]\otimes m)\Big) = \beta^{Y,B}_{ab}\Big([G_b(\sigma,\eta),\omega]\otimes m\Big) = [G_b(\sigma,\eta)]\otimes\check{\mathbb{M}}(\omega)(m) \quad \text{and}$$

$$G_{b\#}\otimes 1\Big(\beta^{IX,IA}_{ab}([\sigma,\eta,\omega]\otimes m)\Big) = G_{b\#}\otimes 1\Big([\sigma,\eta]\otimes\check{\mathbb{M}}(\omega)(m)\Big) = [G_b(\sigma,\eta)]\otimes\check{\mathbb{M}}(\omega)(m), \quad \text{and}$$

therefore $\beta^{Y}_{ab}\circ(G_b\underline{x}1)_{\#} = (G_{b\#}\otimes 1)\circ\beta^{X}_{ab}$.

Recall that $\Delta(1)$ is generated by two 0–simplexes $\{0\}$ and $\{1\}$ in degree zero (see 3.7.1). As in 3.7.2 if $n = 0$ or n is a positive integer we denote by $\{0\}$ (resp. $\{1\}$) the unique n–simplex concentrated at the point 0 (resp. 1) of I. If $c = \{0\}$ or $\{1\}$, then

$$\beta^{IX,IA}_{ab}\big(\hat{\varepsilon}\otimes 1\ (c\otimes[\eta,\omega]\otimes m)\big) = \beta^{IX,IA}_{ab}\big([c,\eta,\omega]\otimes m)\big) = [c,\eta]\otimes\check{\mathbb{M}}(\omega)(m), \text{ and}$$

$$\hat{\varepsilon}\otimes 1\ \big(1\otimes\beta^{X}_{ab}\ (c\otimes[\eta,\omega]\otimes m)\big) = \hat{\varepsilon}\otimes 1\ \big(c\otimes[\eta]\otimes\check{\mathbb{M}}(\omega)(m)\big) = [c,\eta]\otimes\check{\mathbb{M}}(\omega)(m).$$

In degree one $\Delta(1)$ is generated by Ω, and then

$$\beta^{IX,IA}_{ab}(\hat{\varepsilon}\otimes 1(\Omega\otimes[\eta,\omega]\otimes m)) = \beta^{IX,IA}_{ab}\Big(\sum_{k=0}^{n}(-1)^k[(s_n\ldots s_{k+1}s_{k-1}\ldots s_0\Omega,\ s_k\eta,\ s_k\omega)]\otimes m\Big)$$

$= \sum_{k=0}^{n}(-1)^k[(s_n...s_{k+1}s_{k-1}...s_0\Omega, s_k\eta)]\otimes\check{M}(s_k\omega)(m)$, and $\hat{\varepsilon}\otimes 1\left(1\otimes\beta^X_{ab}(\Omega\otimes[\eta,\omega]\otimes m)\right) =$

$\hat{\varepsilon}\otimes 1\ \left(\Omega\otimes[\eta]\otimes\check{M}(\omega)(m)\right) = \left[\sum_{k=0}^{n}(-1)^k[(s_n...s_{k+1}s_{k-1}...s_0\Omega,\ s_k\eta)]\right]\otimes\check{M}(\omega)(m).$

Since $\operatorname{image}(s_k\eta) \subseteq \operatorname{image}(\eta)$, $\check{M}(s_k\eta) = \check{M}(\eta)$, and therefore $\beta^{IX,IA}_{ab}\left(\hat{\varepsilon}\otimes 1(\Omega\otimes[\eta,\omega]\otimes m)\right) = \hat{\varepsilon}\otimes 1\ \left(1\otimes\beta^X_{ab}\ (\Omega\otimes[\eta,\ \omega]\otimes m)\right)$. Thus $\beta^{IX,IA}_{ab}\circ\hat{\varepsilon}\otimes 1 = \hat{\varepsilon}\otimes 1\circ 1\otimes\beta^X_{ab}$, as required.

3.7.12 Recall that $G_{b\#}\circ\hat{\varepsilon}{:}\bar{\Delta}(1)\otimes S_\#(X_b,A_b) \longrightarrow S_\#(Y_b,B_b)$ is a homotopy between $f^0_{b\#}$ and $f^1_{b\#}$, and $(G_{b\#}\underline{x}1)\circ\hat{\varepsilon}{:}\bar{\Delta}(1)\otimes S_\#((X_b,A_b)\underline{x}J(a,b)) \longrightarrow S_\#((Y_b,B_b)\underline{x}J(a,b))$ is a homotopy between $(f^0_b\underline{x}1_{J(a,b)})_\#$ and $(f^1_b\underline{x}1_{J(a,b)})_\#$. Lemma 3.7.11 gives us the following commutative diagram which summarize relations between all these chain maps and $\beta^{X,A}_{ab}$ and $\beta^{Y,B}_{ab}$.

$$
\begin{tikzcd}
S_\#((Y_b,B_b)\underline{x}J(a,b))\otimes M_a \arrow[r, "\beta^{Y,B}_{ab}"] & S_\#(Y_b,B_b)\otimes M_b \\
\bar{\Delta}(1)\otimes S_\#((X_b,A_b)\underline{x}J(a,b))\otimes M_a \arrow[u, "((G_b\underline{x}1)_\#\circ\hat{\varepsilon})\otimes 1"'] \arrow[r, "1\otimes\beta^{X,A}_{ab}"'] & \bar{\Delta}(1)\otimes S_\#(X_b,A_b)\otimes M_b \arrow[u, "(G_{b\#}\circ\hat{\varepsilon})\otimes 1"] \\
S_\#((X_b,A_b)\underline{x}J(a,b))\otimes M_a \arrow[u, "\hat{i}_k"] \arrow[r, "\beta^{X,A}_{ab}"] \arrow[uu, bend left=60, "(f^k_b\underline{x}1)_\#\otimes 1"] & S_\#(X_b,A_b)\otimes M_b \arrow[u, "\hat{i}_k"'] \arrow[uu, bend right=60, "f^k_{b\#}\otimes 1"']
\end{tikzcd},
$$

3.7.13 Proposition (for α's and β's). The following diagram commutes for $k = 0,1$

$$
\begin{tikzcd}
\sum\limits_{a,b} S_\#((Y_b,B_b)\underline{\times}J(a,b))\otimes M_a \arrow[r, shift left, "\alpha^{Y,B}"] \arrow[r, shift right, "\beta^{Y,B}"'] & \sum\limits_{c} S_\#(Y_c,B_c)\otimes M_c \\
\bar{\Delta}(1)\otimes\left[\sum\limits_{a,b} S_\#((X_b,A_b)\underline{\times}J(a,b))\otimes M_a\right] \arrow[u, "((G_b\underline{\times}1)_\#\circ\hat{\varepsilon})\otimes 1"] \arrow[r, shift left, "1\otimes\alpha^{X,A}"] \arrow[r, shift right, "1\otimes\beta^{X,A}"'] & \bar{\Delta}(1)\otimes\left[\sum\limits_{c} S_\#(X_c,A_c)\otimes M_c\right] \arrow[u, "(G_{c\#}\circ\hat{\varepsilon})\otimes 1"'] \\
\sum\limits_{a,b} S_\#((X_b,A_b)\underline{\times}J(a,b))\otimes M_a \arrow[u, "\hat{i}_k"'] \arrow[uu, bend left=70, "((f^k_b\underline{\times}1)_\#\otimes 1)"] \arrow[r, shift left, "\alpha^{X,A}"] \arrow[r, shift right, "\beta^{X,A}"'] & \sum\limits_{c} S_\#(X_c,A_c)\otimes M_c \arrow[u, "\hat{i}_k"] \arrow[uu, bend right=70, "(f^k_{c\#}\otimes 1)"']
\end{tikzcd},
$$

Proof. The claim follows immediately by combining diagrams from 3.7.10 and 3.7.12.

3.7.14 Proof of 3.7.5. We obtained in 3.7.13 the existence of a pair

$$\left(\left((G_{b\#}\underline{\times}1_{J(a,b)})\circ\hat{\varepsilon}\otimes 1_{M_a}\right)_{a,b\,\in\,\mathrm{ob}\,J},\ \left(G_{c\#}\circ\hat{\varepsilon}\otimes 1_{M_c}\right)_{c\,\in\,\mathrm{ob}\,J}\right)$$

of homotopies with properties as it is described in 3.7.5.

3.8 Homotopy Axiom for Singular J-Cohomology

In this section we shall prove singular J-cohomology satisfies the homotopy axiom.

3.8.1 Denote by $\bar{\Delta}^{op}(1)$ a cochain complex obtained from $\bar{\Delta}(1)$ by 'reversing arrows, that is, $\bar{\Delta}^{op}(1)$ is a free $\mathbb{Z}$-cochain complex generated by $\{0\}$ and $\{1\}$ in zero dimension, and by Ω in dimension minus one. The only non-zero differential is $\delta\Omega = \{0\} - \{1\}$. Let $\mathbf{A}$ be a cochain complex. The canonical inclusions $\check{i}_0, \check{i}_1 : \mathbf{A} \longrightarrow \bar{\Delta}(1)\otimes\mathbf{A}$ are defined by $a \longrightarrow \{0\}\otimes a$ and $a \longrightarrow \{1\}\otimes a$, respectively. First, given a homotopy $F:f_0 \simeq f_1$, $F:X\underline{x}I \longrightarrow Y$ (in **CGV**) and a module K we shall construct a cochain map $\check{F}:\bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(Y), K) \longrightarrow \underline{\mathrm{Hom}}(S_\#(X), K)$, such that $\check{F}\circ\check{i}_0 = f_0^{\#}$, $\check{F}\circ\check{i}_1 = f_1^{\#}$. Note that the existence of $\check{F}$ is equivalent to the existence of a classical cochain homotopy between $f_0^{\#}$ and $f_1^{\#}:\underline{\mathrm{Hom}}(S_\#(Y), K) \longrightarrow \underline{\mathrm{Hom}}(S_\#(X), K)$ (similarly to **[Baues]** p. 41).

As the reader may expect, the duality principle shall guide us in constructing the induced cochain homotopy. Let $Y \in$ ob **CGV**. Denote Y^I = **CGV**(I, Y) and for $k = 0,1$, let $p_k:Y^I \longrightarrow Y$, $p \longrightarrow p(k)$. If $\sigma:\Delta_n \longrightarrow Y^I$ is a singular n–simplex, denote by $\tilde{\sigma}:I\underline{x}\Delta_n \longrightarrow Y$ the adjoint of σ. We define a cochain map $\check{\eta} = \check{\eta}_Y:\bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(Y), K) \longrightarrow \underline{\mathrm{Hom}}(S_\#(Y^I), K)$ by $\check{\eta}(\omega\otimes f)(\sigma) = f(\tilde{\sigma}_\#(\varepsilon(\omega\otimes 1_{\Delta_n})))$, where $\varepsilon:S_\#(I)\otimes S_\#(\Delta_n) \longrightarrow S_\#(I\underline{x}\Delta_n)$ is the Eilenberg–MacLane chain map (see 3.7.2) and $\omega \in \bar{\Delta}(1) \subseteq S_\#(I)$.

Note that on generators $\{k\}$, $k = 0,1$, $\check{\eta}(\{k\}\otimes f)(\sigma) = f(p_{k\#}(\sigma)) = p_k^{\#}(f)(\sigma)$ ($p_k^{\#}:\underline{\mathrm{Hom}}(S_\#(Y),K) \longrightarrow \underline{\mathrm{Hom}}(S_\#(Y^I),K)$).

3.8.2 Lemma. $\check{\eta}_Y$ is a cochain map.

Proof. We check the claim on generators. Let $f \in \underline{\mathrm{Hom}}(S_\#(Y),K)$ with $|f| = n$ and let $\sigma:\Delta_n \longrightarrow Y^I$ be a singular n–simplex. For $k = 0,1$, $\delta\eta_Y(\{k\}\otimes f)(\sigma) = \eta_Y(\{k\}\otimes f)(\partial\sigma) = f(p_{k\#}(\partial\sigma)) = f(\partial p_{k\#}(\sigma))$ and $\eta_Y(\delta(\{k\}\otimes f))(\sigma) = \eta_Y(\{k\}\otimes(\delta f))(\sigma) = (\delta f)(p_{k\#}(\sigma)) = f(\partial p_{k\#}(\sigma))$.

Next $\delta\eta_Y(\Omega\otimes f)(\sigma) = \eta_Y(\Omega\otimes f)(\partial\sigma) = f((\tilde{\partial\sigma})_\#(\varepsilon(\Omega\otimes 1_{\Delta_n})))$. If $\partial\sigma = \sum_{i=0}^{n} \sigma^{(i)}$

$= \sum_{i=0}^{n} \sigma \circ \varepsilon^i$, then each $\widetilde{\sigma^{(i)}} = \tilde{\sigma} \circ (1_I x \varepsilon^i)$, and thus, $\tilde{\partial}\tilde{\sigma} = \sum_{i=0}^{n} \tilde{\sigma}(1_I x \varepsilon^i)$.

Therefore, $\delta\eta_Y(\Omega \otimes f)(\sigma) = \sum_{i=0}^{n} f\big((\tilde{\sigma}(1_I x \varepsilon^i))_{\#}(\varepsilon(\Omega \otimes 1_{\Delta_n}))\big)$. Also

$$\eta_Y(\delta(\Omega \otimes f))(\sigma) = \eta_Y(\{0\} \otimes f - \{1\} \otimes f - \Omega \otimes \delta f)(\sigma) =$$

$$f\big(p_{0\#}(\sigma)\big) - f\big(p_{1\#}(\sigma)\big) - (\delta f)\big(\tilde{\sigma}_{\#}(\varepsilon(\omega \otimes 1_{\Delta_n}))\big) =$$

$$f\big(p_{0\#}(\sigma)\big) - f\big(p_{1\#}(\sigma)\big) - f\big(\partial\big(\tilde{\sigma}_{\#}(\varepsilon(\omega \otimes 1_{\Delta_n}))\big)\big) =$$

$$f\big(p_{0\#}(\sigma)\big) - f\big(p_{1\#}(\sigma)\big) - f\big(\tilde{\sigma}_{\#}(\varepsilon\partial(\omega \otimes 1_{\Delta_n}))\big) =$$

$$f\big(p_{0\#}(\sigma)\big) - f\big(p_{1\#}(\sigma)\big) - f\big(\tilde{\sigma}_{\#}(\varepsilon(\{0\} \otimes 1_{\Delta_n} - \{1\} \otimes 1_{\Delta_n} - \Omega \otimes (\sum_{i=0}^{n} \varepsilon^i)))\big) =$$

$$\sum_{i=0}^{n} f\big(\tilde{\sigma}_{\#}(\varepsilon(\Omega \otimes \varepsilon^i))\big).$$

We use the following notation as in [May 2]:

(1) ε^i is a singular simplex induced by the standard k-face operator on Δ_n

(2) s_k is the k-degeneracy operator on Δ_n and "s_k" is the induced singular simplex on Δ_n.

To prove the assertion we shall show that $(1_I x \varepsilon^i)_{\#}\big(\varepsilon(\Omega \otimes 1_{\Delta_n})\big) = \varepsilon(\Omega \otimes \varepsilon^i)$, $i = 0,1...n$. From 3.7.1 we have

$\hat{\varepsilon}(\Omega \otimes 1_{\Delta_n}) = \sum_{k=0}^{n} (-1)^k (s_n ... s_{k+1} s_{k-1} ... s_0 \Omega,\ s_k 1_{\Delta_n})$. Then $(1_I x \varepsilon^i)_{\#}\big(\varepsilon(\Omega \otimes 1_{\Delta_n})\big) =$

$$(1_I x \varepsilon^i)_{\#}\Big[\sum_{k=0}^{n} (-1)^k (s_n ... s_{k+1} s_{k-1} ... s_0 \Omega,\ s_k 1_{\Delta_n})\Big] =$$

$$\sum_{k=0}^{n} (-1)^k (1_I x \varepsilon^i)_{\#} (s_n ... s_{k+1} s_{k-1} ... s_0 \Omega,\ \text{"}s_k\text{"}) =$$

$$\sum_{k=0}^{n} (-1)^k (s_n ... s_{k+1} s_{k-1} ... s_0 \Omega,\quad \varepsilon^i \circ \text{"}s_k\text{"}) \quad = \quad \sum_{k=0}^{n} (-1)^k (s_n ... s_{k+1} s_{k-1} ... s_0 \Omega,\quad s_k \varepsilon^i)$$

$= \hat{\varepsilon}(\Omega\otimes 1_{\Delta_n})$.

3.8.3 Lemma.

(1) The canonical inclusions $\check{i}_0, \check{i}_1 : \underline{\mathrm{Hom}}(S_\#(Y), K) \longrightarrow \bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(Y), K)$ are related to the top and bottom projections $p_0, p_1 : Y^I \longrightarrow Y$, by

$\check{\eta}\circ\check{i}_0 = p_0{}^\#, \qquad \check{\eta}\circ\check{i}_1 = p_1{}^\#.$

(2) The chain $\check{\eta}_Y$ is natural in Y, that is, if $h: Y \longrightarrow Z$ is a continuous function between k-spaces Y and Z, then the following diagram commutes.

$$\begin{tikzcd} \bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(Y), K) \arrow[r, "\hat{\eta}_Y"] & \underline{\mathrm{Hom}}(S_\#(Y^I), K) \\ \bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(Z), K) \arrow[u, "1\otimes h^\#"] \arrow[r, "\hat{\eta}_Z"'] & \underline{\mathrm{Hom}}(S_\#(Z^I), K) \arrow[u, "(h^I)^\#"'] \end{tikzcd}$$

where $h^I: Y^I \longrightarrow Z^I$ is induced by h.

Let $F: f_0 \simeq f_1$, $F: X\underline{x}I \longrightarrow Y$ be a homotopy and $\tilde{F}: X \longrightarrow Y^I$ be the adjoint map. Note that $p_k \circ \tilde{F} = f_k$, $k = 0,1$.

3.8.4 Definition. Define $\check{F}: \bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(Y), K) \longrightarrow \underline{\mathrm{Hom}}(S_\#(X), K)$ by $\check{F} = \tilde{F}^\#\circ\check{\eta}_Y$. Note that $\check{F}\circ\check{i}_k = \tilde{F}^\#\circ\check{\eta}_Y\circ\check{i}_k = \tilde{F}^\#\circ p_k{}^\# = f_k{}^\#$, $k = 0,1$, and therefore $\check{F}$ is a homotopy between $f_0{}^\#$ and $f_1{}^\#$.

3.8.5 Let $f^0, f^1 : (X,A) \longrightarrow (Y,B)$ be two homotopic morphisms of saturated J-pairs with a homotopy $G:(IX,IA) \longrightarrow (Y,B)$, $G: f^0 \simeq f^1$. Recall that f^0

gives rise to a pair of cochain maps $(\check{f}^0_1, \check{f}^0_2)$ such that the right hand square of the following diagram

$$\begin{array}{ccccc}
C^*_{\mathcal{J}}(X,A;\ \mathbb{K}) & \longrightarrow \prod\limits_a \underline{\mathrm{Hom}}(S_\#(X_a,A_a),\mathbb{K}_a) & \xrightarrow{\Psi^{X,A}} & \prod\limits_{b,c} \underline{\mathrm{Hom}}(S_\#((X_c,A_c)\underline{x}\mathcal{J}(b,c)),\mathbb{K}_b) \\
\uparrow{\scriptstyle f^{0\#}} & \uparrow{\scriptstyle \check{f}^0_1} & & \uparrow{\scriptstyle \check{f}^0_2} \\
C^*_{\mathcal{J}}(Y,B;\ \mathbb{K}) & \longrightarrow \prod\limits_a \underline{\mathrm{Hom}}(S_\#(Y_a,B_a),\mathbb{K}_a) & \xrightarrow[\Psi^{Y,B}]{} & \prod\limits_{b,c} \underline{\mathrm{Hom}}(S_\#((Y_c,B_c)\underline{x}\mathcal{J}(b,c)),\mathbb{K}_b)
\end{array}$$

commutes, and the cochain map $f^0_\#:C^*_{\mathcal{J}}(Y,B;\ \mathbb{K}) \longrightarrow C^*_{\mathcal{J}}(X,A;\ \mathbb{K})$, a map between kernels $\ker(\Phi^{X,A}) = C^*_{\mathcal{J}}(X,A;\ \mathbb{K})$ and $\ker(\Phi^{Y,B}) = C^*_{\mathcal{J}}(Y,B;\ \mathbb{K})$, is induced by $(\check{f}^0_1, \check{f}^0_2)$. Similarly, f^1 gives rise to a pair $(\check{f}^1_1, \check{f}^1_2)$ and the induced chain map $f^{1\#}:C^*_{\mathcal{J}}(Y,B;\ \mathbb{K}) \longrightarrow C^*_{\mathcal{J}}(X,A;\ \mathbb{K})$ with the same properties as just described in the case of f^0.

We shall show the following proposition.

3.8.6 Proposition. There exists a pair $(\check{G}_1,\check{G}_2)$ of cochain homotopies, where $\check{G}_1:\check{f}^1_1 \simeq \check{f}^2_1$, $\check{G}_2:\check{f}^1_2 \simeq \check{f}^2_2$, such that for $k = 0,1$ the following diagram commutes:

$$
\begin{tikzcd}
\prod_{a} \underline{\mathrm{Hom}}(S_{\#}(X_a, A_a), K_a) \arrow[r, "\Psi^{X,A}"] & \prod_{b,c} \underline{\mathrm{Hom}}(S_{\#}((X_c, A_c) \underline{\times} \mathcal{J}(b,c)), K_b) \\
\bar{\Delta}(1) \otimes \left[\prod_{a} \underline{\mathrm{Hom}}(S_{\#}(Y_a, B_a), K_a) \right] \arrow[r, "1 \otimes \Psi^{Y,B}"'] \arrow[u, "\check{G}_1"] & \bar{\Delta}(1) \otimes \left[\prod_{b,c} \underline{\mathrm{Hom}}(S_{\#}((Y_c, B_c) \underline{\times} \mathcal{J}(b,c)), K_b) \right] \arrow[u, "\check{G}_2"'] \\
\prod_{a} \underline{\mathrm{Hom}}(S_{\#}(Y_a, B_a), K_a) \arrow[r, "\Psi^{Y,B}"'] \arrow[u, "\check{i}_k"] \arrow[uu, bend left=60, "\check{f}^k_1"] & \prod_{b,c} \underline{\mathrm{Hom}}(S_{\#}((Y_c, B_c) \underline{\times} \mathcal{J}(b,c)), K_b) \arrow[u, "\check{i}_k"'] \arrow[uu, bend right=60, "\check{f}^k_2"']
\end{tikzcd}
$$

Proof. The proof is given in 3.8.14.

3.8.7 Proposition. If $f^0, f^1 : (X,A) \longrightarrow (Y,B)$ are two homotopic morphisms of $\mathcal{J}$-pairs, then the induced chain maps $f^0_{\#}, f^1_{\#} : C^{\mathcal{J}}_{\bullet}(X,A;\ \mathbb{M}) \longrightarrow C^{\mathcal{J}}_{\bullet}(Y,B;\ \mathbb{M})$ are homotopic.

Proof. By the Proposition 3.8.6 there is a pair $(\check{G}_1, \check{G}_2)$ of homotopies. Then the commutativity of the upper middle square induces the existence of a chain map $\check{G} : \ker(1 \otimes \Psi^{Y,B}) \longrightarrow \ker(\Psi^{X,A})$.

Note that $\bar{\Delta}^{op}(1)$ is a free cochain complex and therefore a functor $\bar{\Delta}^{op}(1) \otimes_{\mathbb{Z}} -$ is exact. Thus $\ker(1_{\bar{\Delta}(1)^{op}} \otimes_{\mathbb{Z}} \Psi^{Y,B}) = \bar{\Delta}(1) \otimes_{\mathbb{Z}} \ker(\Psi^{Y,B})$ and it is easy to check that $\check{G}$ is a cochain homotopy between $f^{1\#}$ and $f^{2\#}$.

From Proposition 3.8.7 we have an immediate corollary.

3.8.8 Theorem (HOMOTOPY). If $f^0,f^1:(X,A) \longrightarrow (Y,B)$ are two homotopic morphisms of J-pairs, then $f^{0*} = f^{1*}:H^*_J(Y,B;\ \mathbb{K}) \longrightarrow H^*_J(X,A;\ \mathbb{K})$.

The remainder of this section is devoted to the proof of Proposition 3.8.6. First we obtain a commutative diagram for ϕ's.

3.8.9 Lemma. Let $f^0,f^1:(X,A) \longrightarrow (Y,B)$ be morphisms of J-pairs such that there is a homotopy $G:f^0 \simeq f^1$, $G:(IX,IA) \longrightarrow (Y,B)$. Let $a,b \in$ ob J. The following diagram commutes.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(S_\#(X_a,A_a),\mathbb{K}_a) \arrow[r, "(\tilde{X}_{ab},\tilde{Y}_{ab})^\#"] & \underline{\mathrm{Hom}}(S_\#((X_b,A_b)\underline{x}J(a,b)),\mathbb{K}_a) \\
\underline{\mathrm{Hom}}(S_\#(Y_a,B_a)^I,\mathbb{K}_a) \arrow[r, "((\tilde{Y}_{ab},\tilde{B}_{ab})^I)^\#"'] \arrow[u, "\tilde{G}_a^\#"] & \underline{\mathrm{Hom}}(S_\#((Y_b,B_b)\underline{x}J(a,b))^I,\mathbb{K}_a) . \arrow[u, "(\widetilde{G_b\underline{x}1})^\#"'] \\
\bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(Y_a,B_a),\mathbb{K}_a) \arrow[r, "1\otimes(\tilde{X}_{ab},\tilde{Y}_{ab})^\#"'] \arrow[u, "\check{\eta}"] & \bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#((Y_b,B_b)\underline{x}J(a,b)),\mathbb{K}_a) \arrow[u, "\check{\eta}"']
\end{tikzcd}$$

Proof. Let $\tilde{G}_a$ and $(\widetilde{G_b\underline{x}1}_{J(a,b)})$ be adjoint maps of G_a and $G_b\underline{x}1_{J(a,b)}$, respectively. By reformulation of the naturality of G expressed as in the proof of 3.7.8 in terms of the adjoint maps, we obtain the following commutative diagram

$$\begin{array}{ccc}
(X_b, A_b)\underline{x}\mathcal{J}(a,b) & \xrightarrow{(\tilde{X}_{ab}, \tilde{Y}_{ab})} & (X_a, A_a) \\
\Big\downarrow (\tilde{G}_b \underline{x}\, \tilde{1}) & & \Big\downarrow \tilde{G}_a \\
\big((Y_b, B_b)\underline{x}\mathcal{J}(a,b)\big)^I & \xrightarrow[(\tilde{Y}_{ab}, \tilde{B}_{ab})^I]{} & (Y_a, B_a)^I
\end{array} .$$

Taking $\underline{\mathrm{Hom}}(\_, \kappa_a)$ of the above diagram gives us the commutativity of the upper square. The commutativity of the bottom part follows from 3.8.3.

3.8.10 Lemma (for ϕ's).

(1) The cochain map $\tilde{G}_a^{\#} \circ \check{\eta}: \bar{\Delta}(1) \otimes \underline{\mathrm{Hom}}(S_{\#}(Y_a, B_a), \kappa_a) \longrightarrow \underline{\mathrm{Hom}}(S_{\#}(X_a, A_a), \kappa_a)$ is a homotopy between $f_b^{0\#}$ and $f_b^{1\#}$.

(2) The cochain map $(\widetilde{G_{b\#} \underline{x}\, 1}_{J(a,b)}) \circ \check{\eta}: \bar{\Delta}(1) \otimes \underline{\mathrm{Hom}}(S_{\#}((Y_b, B_b)\underline{x}\mathcal{J}(a,b)), \kappa_a) \longrightarrow \underline{\mathrm{Hom}}(S_{\#}((X_b, A_b)\underline{x}\mathcal{J}(a,b)), \kappa_a)$ is a homotopy between $(f_b^0 \underline{x}\, 1_{J(a,b)})^{\#}$ and $(f_b^1 \underline{x}\, 1_{J(a,b)})^{\#}$.

(3) For $k = 0,1$, the following commutative diagram summarizes relations between all these cochain maps and $\phi_{ab}^{X,A} = (\tilde{X}_{ab}, \tilde{A}_{ab})^{\#}$ and $\phi_{ab}^{Y,B} = (\tilde{Y}_{ab}, \tilde{B}_{ab})^{\#}$.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(S_{\#}(X_a,A_a),K_a) \arrow[r, "\phi^{X,A}_{ab}"] & \underline{\mathrm{Hom}}(S_{\#}((X_b,A_b)\,\underline{x}\,J(a,b)),K_a) \\
\bar\Delta^{op}(1)\otimes\underline{\mathrm{Hom}}(S_{\#}(Y_a,B_a),K_a) \arrow[u, "\tilde G_a^{\#}\circ\check\eta"] \arrow[r, "1\otimes\phi^{Y,B}_{ab}"'] & \bar\Delta^{op}(1)\otimes\underline{\mathrm{Hom}}(S_{\#}((Y_b,B_b)\underline{x}J(a,b)),K_a) \arrow[u, "(\tilde G_b\underline{x}1)^{\#}\circ\check\eta"'] \\
\underline{\mathrm{Hom}}(S_{\#}(Y_a,B_a),K_a) \arrow[u, "\check i_k"] \arrow[uu, bend left=60, "(\check f^k_b)^{\#}"] \arrow[r, "\phi^{Y,B}_{ab}"'] & \underline{\mathrm{Hom}}(S_{\#}((Y_b,B_b)\,\underline{x}\,J(a,b)),K_a) \arrow[u, "\check i_k"'] \arrow[uu, bend right=60, "(f^k_b\underline{x}1)^{\#}"']
\end{tikzcd}$$

3.8.11 Definition. Let a,b $\in$ ob J. Define

$$\psi^{(X,A)^I}_{ab}:\underline{\mathrm{Hom}}(S_{\#}((X_b,A_b)^I),K_b) \longrightarrow \underline{\mathrm{Hom}}(S_{\#}(((X_b,A_b)\underline{x}J(a,b))^I),K_a),$$

by $\psi^{(X,A)^I}_{ab}(f)_{p,q}(\sigma,\eta) = \check K(\eta(0))(f_{p,q}(\sigma))$ for $f = \{f_{p,q}:S_p((X_b,A_b)^I) \longrightarrow K_b\}$. As in 3.4.5 we have that $\psi^{(X,A)^I}_{ab}$ is a cochain map.

3.8.12 Lemma. Let $f^0,f^1:(X,A) \longrightarrow (Y,B)$ be morphisms of J-pairs such that there is a homotopy $G:f^0 \simeq f^1$, $G:(IX,IA) \longrightarrow (Y,B)$. Let a,b $\in$ ob J. The following diagram commutes.

$$\begin{array}{ccc}
\underline{\mathrm{Hom}}(S_\#(X_b, A_b), K_b) & \xrightarrow{\psi^{X,A}_{ab}} & \underline{\mathrm{Hom}}(S_\#((X_b, A_b)\underline{x}J(a,b)), K_a) \\
\uparrow \tilde{G}_b{}^\# & & \uparrow (\tilde{G}_b\underline{x}\tilde{1})^\# \\
\underline{\mathrm{Hom}}(S_\#(Y_b, B_b)^I, K_b) & \xrightarrow{\psi^{(Y,B)}_{ab}} & \underline{\mathrm{Hom}}(S_\#((Y_b, B_b)\underline{x}J(a,b))^I, K_a)\,. \\
\uparrow \check{\eta} & & \uparrow \check{\eta} \\
\bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(Y_b, B_b), K_b) & \xrightarrow[1\otimes\psi^{Y,B}_{ab}]{} & \bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#((Y_b, B_b)\underline{x}J(a,b)), K_a)
\end{array}$$

Proof. We show that both squares commute by the checking the generators directly. The upper square commutes since

$\psi^{X,A}_{ab}(\tilde{G}_b{}^\#(f))_{p,q}(\sigma,\eta) = \check{K}(\eta)(\tilde{G}_b{}^\#(f)_{p,q}(\sigma)) = \check{K}(\eta)\left[f_{p,q}(\tilde{G}_b{}^\#(\sigma))\right]$, and $(\tilde{G}_b\underline{x}\tilde{1}_{J(a,b)})^\#\left[\psi^{(Y,B)^I}_{ab}(f)\right]_{p,q}(\sigma,\eta) = \left[\psi^{(Y,B)^I}_{ab}(f)\right]_{p,q}((\tilde{G}_b\underline{x}\tilde{1}_{J(a,b)})_\#(\sigma,\eta)) = \left[\psi^{(Y,B)^I}_{ab}(f)\right]_{p,q}((\tilde{G}_{b\#}(\sigma),\eta^I)$, where $\eta^I{:}\Delta_n \longrightarrow J(a,b)^I$ is given by $\eta^I(x)(t) = \eta(x)$.

Continuing, we have $(\tilde{G}_b\underline{x}\tilde{1}_{J(a,b)})^\#\left[\psi^{(Y,B)^I}_{ab}(f)\right]_{p,q}(\sigma,\eta) = \check{K}(\eta)\left[f_{p,q}(\tilde{G}_{b\#}(\sigma))\right]$ as required. The commutativity of the bottom square follows from the following calculation; first observe that $\psi^{(Y,B)^I}_{ab}(\check{\eta}(\omega\otimes f))_{p,q}(\sigma_I,\eta_I) = \check{K}(\eta_{I(0)})\left[\check{\eta}(\omega\otimes f)_{p,q}(\sigma_I)\right]$ and that $\check{\eta}\left[(1\otimes\psi^{Y,B}_{ab})(\omega\otimes f)\right]_{p,q}(\sigma_I,\eta_I) = \check{\eta}\left[\omega\otimes\psi^{Y,B}_{ab}(f)\right]_{p,q}(\sigma_I,\eta_I)$. If $\omega = \{k\}$, $k = 0,1,$

then $\psi^{(Y,B)^I}_{ab}\left(\check{\eta}(\{k\}\otimes f)\right)_{p,q}(\sigma_I,\eta_I) = \check{\kappa}(\eta_{I(0)})\left[f_{p,q}\left(p_{k\#}(\sigma_I)\right)\right]$ and $\check{\eta}\left[(1\otimes\psi^{Y,B}_{ab})(\{k\}\otimes f)\right]_{p,q}(\sigma_I,\eta_I) = \psi^{Y,B}_{ab}(f)_{p,q}\left(p_{k\#}(\sigma_I),p_{k\#}(\eta_I)\right) = \check{\kappa}(p_{k\#}(\eta_I))\left[f_{p,q}\left(p_{k\#}(\sigma_I)\right)\right]$. Equality holds since both of the images of $\eta_{I(0)}$ and $p_{k\#}(\eta_I)$ lie in the same path component and thus $\check{\kappa}(\eta_{I(0)}) = \check{\kappa}(p_{k\#}(\eta_I)(0))$.

If $\omega = \Omega$, then $\psi^{(Y,B)^I}_{ab}\left(\check{\eta}(\Omega\otimes f)\right)_{p,q}(\sigma_I,\eta_I) = \check{\kappa}(\eta_{I(0)})\left[f_{p,q}\left(\tilde{\sigma}_{I\#}(\varepsilon(\Omega\otimes 1_{\Delta_n}))\right)\right]$ and $\check{\eta}\left[(1\otimes\psi^{Y,B}_{ab})(\Omega\otimes f)\right]_{p,q}(\sigma_I,\eta_I) = \psi^{Y,B}_{ab}(f)_{p,q}\left[(\widetilde{\sigma_I,\eta_I})_{\#}(\varepsilon(\Omega\otimes 1_{\Delta_n}))\right] = \psi^{Y,B}_{ab}(f)_{p,q}\left[\tilde{\sigma}_{I\#}(\varepsilon(\Omega\otimes 1_{\Delta_n})),\tilde{\eta}_{I\#}(\varepsilon(\Omega\otimes 1_{\Delta_n}))\right] = \check{\kappa}\left(\tilde{\eta}_{I\#}(\varepsilon(\Omega\otimes 1_{\Delta_n}))\right)\left[f_{p,q}\left(\tilde{\sigma}_{I\#}(\varepsilon(\Omega\otimes 1_{\Delta_n}))\right)\right]$, and equality holds since both of the images of $\eta_{I(0)}$ and $\tilde{\eta}_{I\#}(\varepsilon(\Omega\otimes 1_{\Delta_n}))$ lie in the same path component and thus $\check{\kappa}(\eta_{I(0)}) = \check{\kappa}\left(\tilde{\eta}_{I\#}(\varepsilon(\Omega\otimes 1_{\Delta_n}))\right)$.

3.8.13 Summary for ψ. For $k = 0,1$ the following commutative diagram summarize relations between $\tilde{G}_b^{\#}$, $(\widetilde{G_b \underline{x} 1}_{J(a,b)})^{\#}$, $\check{\eta}$ and $\psi^{X,A}_{ab}$ with $\psi^{Y,B}_{ab}$.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\mathcal{S}_\#(X_b,A_b),\mathbb{K}_b) \arrow[r, "\psi^{X,A}_{ab}"] & \underline{\mathrm{Hom}}(\mathcal{S}_\#((X_b,A_b)\,\underline{x}\,\mathcal{J}(a,b)),\mathbb{K}_a) \\
\bar\Delta^{op}(1)\otimes\underline{\mathrm{Hom}}(\mathcal{S}_\#(Y_b,B_b),\mathbb{K}_b) \arrow[r, "1\otimes\psi^{Y,B}_{ab}"'] \arrow[u, "\tilde G_b{}^\#\circ\check\eta"] & \bar\Delta^{op}(1)\otimes\underline{\mathrm{Hom}}(\mathcal{S}_\#((Y_b,B_b)\underline{x}\mathcal{J}(a,b)),\mathbb{K}_a) \arrow[u, "(\tilde G_b\underline{x}\tilde 1)^\#\circ\check\eta"'] \\
\underline{\mathrm{Hom}}(\mathcal{S}_\#(Y_b,B_b),\mathbb{K}_b) \arrow[r, "\psi^{Y,B}_{ab}"'] \arrow[u, "\overset{Y}{i}_k"] \arrow[uu, bend left=70, "(\check f^k_b)^\#"] & \underline{\mathrm{Hom}}(\mathcal{S}_\#((Y_b,B_b)\,\underline{x}\,\mathcal{J}(a,b)),\mathbb{K}_a) \arrow[u, "\overset{Y}{i}_k"'] \arrow[uu, bend right=70, "(f^k_b\underline{x}1)^\#"']
\end{tikzcd}$$

3.8.14 Definition. Define

$$\check G_1 = (\tilde G_a{}^\#\circ\check\eta)_{a\,\in\,\mathrm{ob}\,\mathcal{J}},\quad \check G_2 = ((\tilde G_b\underline{x}1_{\mathcal{J}(a,b)})^\#\circ\check\eta)_{a,b\,\in\,\mathrm{ob}\,\mathcal{J}}.$$

Note that $\check G_1$ is a cochain homotopy between $\check f^1_1$ and $\check f^2_1$, and $\check G_2$ is a cochain homotopy between $\check f^1_2$ and $\check f^2_2$ (see 3.8.6).

3.8.15 Proof of 3.8.6. The claim follows by combining 3.8.10, 3.8.13 and 3.8.14.

3.9 The Excision Axiom for $\mathcal{J}$-Homology

3.9.1 Let Z be a topological space. Let $\mathrm{sd}_Z:\mathcal{S}_\#(Z) \longrightarrow \mathcal{S}_\#(Z)$ denote the classical subdivision functor and $D_Z:\mathcal{S}_\#(Z) \longrightarrow \mathcal{S}_\#(Z)$ (degree +1) denote the chain homotopy from sd_Z to 1 ([Spanier] p.177). Recall that both sd_Z and D_Z are functorial in Z; that is, if $f:Z \longrightarrow W$ is a map of topological spaces, then there are commutative squares

$$\begin{array}{ccc} S_\#(W) & \xrightarrow{sd_W} & S_\#(W) \\ {\scriptstyle f_\#}\uparrow & & \uparrow{\scriptstyle f_\#} \\ S_\#(Z) & \xrightarrow[sd_Z]{} & S_\#(Z) \end{array} \qquad\qquad \begin{array}{ccc} S_\#(W) & \xrightarrow{D_W} & S_\#(W) \\ {\scriptstyle f_\#}\uparrow & & \uparrow{\scriptstyle f_\#} \\ S_\#(Z) & \xrightarrow[D_Z]{} & S_\#(Z) \end{array}$$

Let n be an nonnegative integer. Let σ be a singular n-simplex on X. Note that by definition $sd(\sigma) = \sigma_\#(sd(1_{\Delta_n}))$ (see [Spanier] p. 177). We can write $sd(1_{\Delta_n}) = \sum_{k=0}^{N(n)} z_k\ \omega_k$ for some singular n-simplexes ω_k of Δ_n and integers $z_k \in \{-1, +1\}$, which is independent of σ or the space X, but is a function of n only. Therefore for every σ, $sd(\sigma) = \sum_{k=0}^{N(n)} z_k\ \sigma\circ\omega_k$.

Also by definition, $D(\sigma) = \sigma_\#(D(1_{\Delta_n}))$, and we have $D(1_{\Delta_n}) = \sum_{k=0}^{M(n)} \bar{z}_k\ \bar{\omega}_k$ for some singular (n+1)-simplexes $\bar{\omega}_k$ of Δ_n and integers $\bar{z}_k \in \{-1, +1\}$, Therefore for every σ, $D(\sigma) = \sum_{k=0}^{M(n)} \bar{z}_k\ \sigma\circ\bar{\omega}_k$, where M(n), $\bar{z}_k$, and $\bar{\omega}_k$ depend on n only.

Recall that $\tilde{D} = \tilde{D}_Z:\bar{\Delta}(1)\otimes S_\#(Z) \longrightarrow S_\#(Z)$ defined by

$\tilde{D}(\{0\}\otimes\sigma) = id_Z(\sigma) = \sigma$, $\tilde{D}(\{1\}\otimes\sigma) = sd_Z(\sigma) = \sum_{k=0}^{N(n)} z_k\ \sigma\circ\omega_k$, $|\sigma| = n$, and

$\tilde{D}(\Omega\otimes\sigma) = D_Z(\sigma) = \sum_{k=0}^{M(n)} \bar{z}_k\ \sigma\circ\bar{\omega}_k$, $|\sigma| = n$, corresponds to the chain homotopy D.

$\tilde{D}$ is also called a homotopy.

Then $\tilde{D}_Z$ is functorial in Z and it is expressed by the following commutative square:

$$\begin{tikzcd} \bar{\Delta}(1)\otimes S_\#(W) \arrow[r, "\tilde{D}_W"] & S_\#(W) \\ \bar{\Delta}(1)\otimes S_\#(Z) \arrow[r, "\tilde{D}_Z"'] \arrow[u, "1\otimes f_\#"] & S_\#(Z) \arrow[u, "f_\#"'] \end{tikzcd}$$

Now, we shall establish that *sd* induces a chain map on $C^J_*(X;\ \mathbb{M})$.

3.9.2 Lemma (for α's). The following diagram commutes.

$$\begin{tikzcd} S_\#\left(X_b \underline{\times} J(a,b)\right)\otimes \mathbb{M}_a \arrow[r, "\alpha^X_{ab}"] & S_\#(X_a)\otimes \mathbb{M}_a \\ S_\#\left(X_b \underline{\times} J(a,b)\right)\otimes \mathbb{M}_a \arrow[r, "\alpha^X_{ab}"'] \arrow[u, "sd_{X_b\times J(a,b)}\otimes 1_{\mathbb{M}_a}"] & S_\#(X_a)\otimes \mathbb{M}_a \arrow[u, "sd_{X_a}\otimes 1_{\mathbb{M}_a}"'] \end{tikzcd}$$

Proof. By the functorial property of the functor *sd*, the following diagram commutes.

$$\begin{tikzcd} S_\#(X_b \underline{x} J(a,b)) \arrow[r, "\tilde{X}_{ab\#}"] & S_\#(X_a) \\ S_\#(X_b \underline{x} J(a,b)) \arrow[u, "sd_{XbxJ(a,b)}"] \arrow[r, "\tilde{X}_{ab\#}"'] & S_\#(X_a) \arrow[u, "sd_{X_a}"'] \end{tikzcd}$$

Tensoring with $\mathbb{M}_a$ yields the assertion.

3.9.3 Lemma (for β's). The following diagram commutes.

$$\begin{tikzcd} S_\#(X_b \underline{x} J(a,b)) \otimes \mathbb{M}_a \arrow[r, "\beta^X_{ab}"] & S_\#(X_b) \otimes \mathbb{M}_b \\ S_\#(X_b \underline{x} J(a,b)) \otimes \mathbb{M}_a \arrow[u, "sd_{XbxJ(a,b)} \otimes 1_{\mathbb{M}_a}"] \arrow[r, "\beta^X_{ab}"'] & S_\#(X_b) \otimes \mathbb{M}_b \arrow[u, "sd_{X_b} \otimes 1_{\mathbb{M}_b}"'] \end{tikzcd}$$

Proof. By direct calculation we have

$$\beta^X_{ab}\left(sd_{XbxJ(a,b)} \otimes 1_{\mathbb{M}_a}((\sigma,\eta)\otimes m)\right) = \beta^X_{ab}(sd(\sigma,\eta)\otimes m\ ) = \beta^X_{ab}\left(\left(\sum_{k=0}^{N(n)} z_k(\sigma\circ\omega_k,\eta\circ\omega_k)\right)\otimes m\right)$$

$$= \sum_{k=0}^{N(n)} z_k\ \sigma\circ\omega_k \otimes \check{\mathbb{M}}(\eta\circ\omega_k)(m) \quad \text{and} \quad sd_{X_b}\otimes 1_{\mathbb{M}_b}\left(\beta^X_{ab}((\sigma,\eta)\otimes m)\right) = sd_{X_b}\otimes 1_{\mathbb{M}_b}(\sigma\otimes\check{\mathbb{M}}(\eta)(m))$$

$$= sd(\sigma)\otimes\check{\mathbb{M}}(\eta)(m) = \left(\sum_{k=0}^{N(n)} z_k\ \sigma\circ\omega_k\right)\otimes\check{\mathbb{M}}(\eta)(m) = \sum_{k=0}^{N(n)} z_k\ \sigma\circ\omega_k\otimes\check{\mathbb{M}}(\eta)(m).$$

Since image$(\eta\circ\omega_k) \subseteq$ image(η), $\check{\mathbb{M}}(\eta\circ\omega_k) = \check{\mathbb{M}}(\eta)$, and therefore

$sd_{XbxJ(a,)} \otimes 1_{M_a}\left[\beta^X_{ab}((\sigma,\eta)\otimes m)\right] = sd_{Xb} \otimes 1_{M_b}\left[\beta^X_{ab}((\sigma,\eta)\otimes m)\right]$ as required.

3.9.4 Lemma. The functor sd induces the following commutative square of chain maps

$$\begin{array}{ccc}
\sum\limits_{a,b} S_\#(X_b \underline{x} J(a,b)) \otimes M_a & \xrightarrow{\Phi^X} & \sum\limits_{c} S_\#(X_c) \otimes M_c \\
{\scriptstyle (sd_{XbxJ(a,b)} \otimes 1_{M_a})} \uparrow & & \uparrow {\scriptstyle (sd_{Xc} \otimes 1_{M_c})} \\
\sum\limits_{a,b} S_\#(X_b \underline{x} J(a,b)) \otimes M_a & \xrightarrow[\Phi^X]{} & \sum\limits_{c} S_\#(X_c) \otimes M_c
\end{array}$$

Proof. Combine 3.9.2 and 3.9.3 and the definition of Φ.

Denote $sd_1 = \left(sd_{XbxJ(a,b)} \otimes 1_{M_a}\right)_{a,b \in \text{ob } J}$, $sd_2 = \left(sd_{Xc} \otimes 1_{M_c}\right)_{c \in \text{ob } J}$.

The commutative diagram in 3.9.4 gives us the following corollary.

?Corollary. The functor sd induces a chain map $\overline{sd}: C^J_*(X; M) \longrightarrow C^J_*(X; M)$.

Now, we establish that $\tilde{D}$ induces a homotopy on $C^J_*(X; M)$.

3.9.5 Lemma (for α's). Let $a,b \in \text{ob } J$. The following square commutes.

$$\begin{tikzcd}
S_\#(X_b \underline{\times} \mathcal{J}(a,b)) \otimes \mathbb{M}_a \arrow[r, "\alpha^X_{ab}"] & S_\#(X_a) \otimes \mathbb{M}_a \\
\bar{\Delta}(1) \otimes S_\#(X_b \underline{\times} \mathcal{J}(a,b)) \otimes \mathbb{M}_a \arrow[r, "1 \otimes \alpha^X_{ab}"'] \arrow[u, "\tilde{D}_{X_b \times J(a,b)} \otimes 1"] & \bar{\Delta}(1) \otimes S_\#(X_a) \otimes \mathbb{M}_a \arrow[u, "\tilde{D}_{X_a} \otimes 1"']
\end{tikzcd}$$

$\tilde{D}_{X_b \times J(a,b)} \otimes 1_{\mathbb{M}_a}$ is a homotopy between 1 and $sd_{X_b \times J(a,b)} \otimes 1_{\mathbb{M}_a}$. $\tilde{D}_{X_a}$ is a homotopy between 1 and $sd_{X_a} \otimes 1_{\mathbb{M}_a}$.

Proof. By the functoriality property of the functor $\tilde{D}$ (see 3.9.1), the following square commutes.

$$\begin{tikzcd}
S_\#(X_b \underline{\times} \mathcal{J}(a,b)) \arrow[r, "\tilde{X}_{ab\#}"] & S_\#(X_a) \\
\bar{\Delta}(1) \otimes S_\#(X_b \underline{\times} \mathcal{J}(a,b)) \arrow[r, "1 \otimes \tilde{X}_{ab\#}"'] \arrow[u, "\tilde{D}_{X_b \times J(a,b)}"] & \bar{\Delta}(1) \otimes S_\#(X_a) \arrow[u, "\tilde{D}_{X_a}"']
\end{tikzcd}$$

Tensoring with $\mathbb{M}_a$ yields the assertion.

3.9.6 Lemma (for β's). The following diagram commutes.

$$\begin{tikzcd}
\mathcal{S}_{\#}(X_b \times J(a,b)) \otimes \mathbb{M}_a \arrow[r, "\beta^X_{ab}"] & \mathcal{S}_{\#}(X_b) \otimes \mathbb{M}_b \\
\bar{\Delta}(1) \otimes \mathcal{S}_{\#}(X_b \times J(a,b)) \otimes \mathbb{M}_a \arrow[r, "1\otimes\beta^X_{ab}"'] \arrow[u, "\tilde{D}_{X_b \times J(a,b)} \otimes 1_{\mathbb{M}_a}"] & \bar{\Delta}(1) \otimes \mathcal{S}_{\#}(X_b) \otimes \mathbb{M}_b \arrow[u, "\tilde{D}_{X_b} \otimes 1_{\mathbb{M}_b}"']
\end{tikzcd}$$

Proof. Clearly the square commutes for generators of the form $\{0\}\otimes(\sigma,\eta)\otimes m$ since $\tilde{D}(\{0\}\otimes\_) = 1$. Since $\tilde{D}(\{1\}\otimes\_) = sd\_$, the claim for generators of form $\{1\}\otimes(\sigma,\eta)\otimes m$ follows from 3.9.3. Take $\{1\}\otimes(\sigma,\eta)\otimes m$.

Recall $\tilde{D}(\Omega\otimes(\sigma,\eta)) = D(\sigma,\eta) = \sum_{k=0}^{M(n)} \bar{z}_k\,(\sigma,\eta)\circ\bar{\omega}_k$, $|(\sigma,\eta)| = |\sigma| = n$. Then

$$\beta^X_{ab}\left[\tilde{D}_{X_b \times J(a,b)}\otimes 1_{\mathbb{M}_a}(\Omega\otimes(\sigma,\eta)\otimes m)\right] = \beta^X_{ab}(D(\sigma,\eta)\otimes m) = \beta^X_{ab}\left[\left(\sum_{k=0}^{M(n)} \bar{z}_k(\sigma\circ\bar{\omega}_k,\eta\circ\bar{\omega}_k)\right)\otimes m\right]$$

$$= \sum_{k=0}^{M(n)} \bar{z}_k\ \sigma\circ\bar{\omega}_k\otimes\check{\mathbb{M}}(\eta\circ\bar{\omega}_k)(m), \text{ and}$$

$$\tilde{D}_{X_b}\otimes 1_{\mathbb{M}_b}\left[1\otimes\beta^X_{ab}(\Omega\otimes(\sigma,\eta)\otimes m)\right] = \tilde{D}_{X_b}\otimes 1_{\mathbb{M}_b}(\Omega\otimes\sigma\otimes\check{\mathbb{M}}(\eta)(m)) = D(\sigma)\otimes\check{\mathbb{M}}(\eta)(m) =$$

$$\left(\sum_{k=0}^{M(n)} \bar{z}_k\ \sigma\circ\bar{\omega}_k\right)\otimes\check{\mathbb{M}}(\eta)(m) = \sum_{k=0}^{M(n)} \bar{z}_k\ \sigma\circ\bar{\omega}_k\otimes\check{\mathbb{M}}(\eta)(m).$$ Since $\text{image}(\eta\circ\bar{\omega}_k) \subseteq \text{image}(\eta)$, $\check{\mathbb{M}}(\eta\circ\bar{\omega}_k) = \check{\mathbb{M}}(\eta)$ and therefore $\beta^X_{ab}\left[\tilde{D}_{X_b \times J(a,b)}\otimes 1_{\mathbb{M}_a}(\Omega\otimes(\sigma,\eta)\otimes m)\right] = \tilde{D}_{X_b}\otimes 1_{\mathbb{M}_b}\left[1\otimes\beta^X_{ab}(\Omega\otimes(\sigma,\eta)\otimes m)\right]$ as required.

3.9.7 Lemma. $\tilde{D}$ induces the following commutative square of chain maps

$$\begin{array}{ccc}
\sum\limits_{a,b} S_{\#}(X_b \times J(a,b)) \otimes M_a & \xrightarrow{\Phi^X} & \sum\limits_{c} S_{\#}(X_c) \otimes M_c \\
(\tilde{D}_{X_b \times J(a,b)} \otimes 1_{M_a}) \uparrow & & \uparrow (\tilde{D}_{X_c} \otimes 1_{M_c}) \\
\bar{\Delta}(1) \otimes \sum\limits_{a,b} S_{\#}(X_b \times J(a,b)) \otimes M_a & \xrightarrow[1 \otimes \Phi^X]{} & \bar{\Delta}(1) \otimes \sum\limits_{c} S_{\#}(X_c) \otimes M_c
\end{array}$$

Proof. Combine 3.9.5 and 3.9.6 and the definition of Φ's.

3.9.8 Definition. Denote

$$\hat{D}_1 = \left(\tilde{D}_{X_b \times J(a,b)} \otimes 1_{M_a}\right)_{a,b \in \text{ob } J}, \qquad \hat{D}_2 = \left(\tilde{D}_{X_c} \otimes 1_{M_c}\right)_{c \in \text{ob } J}.$$

Note that $\hat{D}_1$ is a homotopy between 1 and sd_1, and $\hat{D}_2$ is a homotopy between 1 and sd_2 (see 3.9.4). The commutative diagram in 3.9.7 gives us the following corollary.

Corollary. $\tilde{D}$ induces a homotopy $\tilde{D}: \bar{\Delta}(1) \otimes C_*^J(X; M) \longrightarrow C_*^J(X; M)$ between $1_{C_*^J(X;M)}$ and $\overline{sd}$.

The next goal is to define chain subcomplex generated by the covering.

3.9.9 Definition. Let $X: J^{op} \longrightarrow$ **CGV** be a J-diagram. Let $\mathcal{U} = \{A\}$ be a collection of saturated J-subdiagrams of X, such that

$$X_a = \bigcup \{\text{int}_{X_a}(A_a) \mid A \in \mathcal{U}\}.$$

The inclusions $A \subseteq X$ are denoted by i^A, and usually all maps or morphisms induced by A will be indexed by the letter A.

(1) Let $a \in$ ob J. Then the family $\mathcal{U}_a = \{A_a |\ A \in \mathcal{U}\}$ is a covering of X_a with property $X_a = \cup \{\mathrm{int}_{X_a}(Z)) |\ Z \in \mathcal{U}_a\}$. Define $S(\mathcal{U}_a)$ to be a chain subcomplex of $S_\#(X_a)$ generated by all singular simplexes of X_a which map into some A_a, $A \in \mathcal{U}$. Denote by $i^{\mathcal{U}}_a$ the inclusion of $S(\mathcal{U}_a)$ into $S_\#(X_a)$. Denote by u^A_a the inclusion of $S_\#(A_a)$ into $S(\mathcal{U}_a)$. Note that $i^A_{a\#} = i^{\mathcal{U}}_a \circ u^A_a$.

(2) Let $a,b \in$ ob J. Then $\mathcal{W}_{ab} = \{A_a \underline{x} J(a,b) |\ A \in \mathcal{U}\}$ is a covering of $X_a \underline{x} J(a,b)$ with property $X_a \underline{x} J(a,b) = \cup \{\mathrm{int}_{X_a x J(a,b)}(Z)) |\ Z \in \mathcal{W}_{ab}\}$. Define $S(\mathcal{W}_{ab})$ to be a chain subcomplex of $S_\#(X_a \underline{x} J(a,b))$ generated by all singular simplexes of $X_a \underline{x} J(a,b)$ which map into some $A_a \underline{x} J(a,b)$, $A \in \mathcal{U}$. Denote by $i^{\mathcal{W}}_{ab}$ the inclusion of $S(\mathcal{W}_{ab})$ into $S_\#(X_a \underline{x} J(a,b))$. Denote by u^A_{ab} the inclusion of $S_\#(A_a \underline{x} J(a,b))$ into $S(\mathcal{W}_{ab})$. Note that

$$(i^A_a \underline{x} 1)_\# = i^{\mathcal{W}}_{ab} \circ u^A_{ab}.$$

Let $a,b \in$ ob J. Recall that $\tilde{X}_{ab\#}: S_\#(X_a \underline{x} J(a,b)) \longrightarrow S_\#(X_a)$, where $\tilde{X}_{ab}$ is the adjoint map of X_{ab}.

3.9.10 Claim. $\tilde{X}_{ab\#}$ maps $S(\mathcal{W}_{ab})$ to $S(\mathcal{U}_a)$, and hence it defines a map $S\mathcal{U}_{ab}: S(\mathcal{W}_{ab}) \longrightarrow S(\mathcal{U}_a)$ which fulfills the following diagram for every $A \in \mathcal{U}$.

$$\begin{tikzcd}
S_\#(X_b \underline{x} J(a,b)) \arrow[r, "\tilde{X}_{ab\#}"] & S_\#(X_a) \\
S(\mathcal{U}\mathcal{J}_{ab}) \arrow[u, "i^{\mathcal{U}\mathcal{J}}_{ab}"'] \arrow[r, dotted, "S\mathcal{U}_{ab}"] & S(\mathcal{U}_a) \arrow[u, "i^{\mathcal{U}}_{a}"] \\
S_\#(A_b \underline{x} J(a,b)) \arrow[u, "u^{A}_{ab}"'] \arrow[r, "\tilde{A}_{ab\#}"'] \arrow[uu, bend left=60, "(i^{A}_{a} \underline{x} 1)_\#"] & S_\#(A_a) \arrow[u, "u^{A}_{a}"] \arrow[uu, bend right=60, "i^{A}_{a\#}"']
\end{tikzcd}$$

Proof. The commutativity of the above diagram of solid arrows follows from definition of involved maps (see 3.9.9). Commutativity of the outer square allows us to define $S\mathcal{U}_{ab}$ as the restriction map of $\tilde{X}_{ab\#}$ to $S(\mathcal{U}\mathcal{J}_{ab})$.

Let a,b $\in$ ob J. Define $\alpha^{\mathcal{U}}_{ab}: S(\mathcal{U}\mathcal{J}_{ab}) \otimes_{\mathbb{Z}} \mathbb{M}_a \longrightarrow S(\mathcal{U}_a) \otimes_{\mathbb{Z}} \mathbb{M}_a$ by $\alpha^{\mathcal{U}}_{ab} = S\mathcal{U}_{ab} \otimes 1_{\mathbb{M}_a}$.

3.9.11 Lemma (for α's).

(1) $\left(i^{\mathcal{U}\mathcal{J}}_{ab} \otimes 1_{\mathbb{M}_a}\right) \circ \left(u^{A}_{ab} \otimes 1_{\mathbb{M}_a}\right) = (i^{A}_{a} \underline{x} 1)_\# \otimes 1_{\mathbb{M}_a}$,

(2) $\left(i^{\mathcal{U}}_{a} \otimes 1_{\mathbb{M}_a}\right) \circ \left(u^{A}_{a} \otimes 1_{\mathbb{M}_a}\right) = i^{A}_{a\#} \otimes 1_{\mathbb{M}_a}$

(3) The following diagram commutes.

$$
\begin{tikzcd}
S_\#(X_b \underline{\times} J(a,b)) \otimes \mathbb{M}_a \arrow[r, "\alpha^X_{ab}"] & S_\#(X_a) \otimes \mathbb{M}_a \\
S(\mathcal{U}J_{ab}) \otimes \mathbb{M}_a \arrow[u, "i^{\mathcal{U}J}_{ab}\otimes 1"] \arrow[r, "\alpha^{\mathcal{U}}_{ab}"] & S(\mathcal{U}_a) \otimes \mathbb{M}_a \arrow[u, "i^{\mathcal{U}}_a\otimes 1"'] \\
S_\#(A_b \underline{\times} J(a,b)) \otimes \mathbb{M}_a \arrow[u, "u^A_{ab}\otimes 1"] \arrow[r, "\alpha^A_{ab}"'] \arrow[uu, bend left=60, "(i^A_b\underline{\times}1)_\#\otimes 1"] & S_\#(A_a) \otimes \mathbb{M}_a \arrow[u, "u^A_a\otimes 1"'] \arrow[uu, bend right=60, "i^A_{a\#}\otimes 1"']
\end{tikzcd}
$$

Proof. Tensoring the diagram in 3.9.10 with $\mathbb{M}_a$ gives us the claim.

Let a,b $\in$ ob J. Define $\beta^{\mathcal{U}}_{ab}: S(\mathcal{U}J_{ab}) \otimes_{\mathbb{Z}} \mathbb{M}_a \longrightarrow S(\mathcal{U}_b) \otimes_{\mathbb{Z}} \mathbb{M}_b$ by $\beta^{\mathcal{U}}_{ab} = \beta^X_{ab} | S(\mathcal{U}J_{ab}) \otimes_{\mathbb{Z}} \mathbb{M}_a$.

3.9.12 Lemma (for β's). The following diagram commutes.

$$
\begin{tikzcd}
S_\#(X_b \underline{\times} J(a,b)) \otimes \mathbb{M}_a \arrow[r, "\beta^X_{ab}"] & S_\#(X_b) \otimes \mathbb{M}_b \\
S(\mathcal{U}J_{ab}) \otimes \mathbb{M}_a \arrow[u, "i^{\mathcal{U}J}_{ab}\otimes 1"] \arrow[r, dotted, "\beta^{\mathcal{U}}_{ab}"] & S(\mathcal{U}_b) \otimes \mathbb{M}_b \arrow[u, "i^{\mathcal{U}}_b\otimes 1"'] \\
S_\#(A_b \underline{\times} J(a,b)) \otimes \mathbb{M}_a \arrow[u, "u^A_{ab}\otimes 1"] \arrow[r, "\beta^A_{ab}"'] \arrow[uu, bend left=60, "(i^A_b\underline{\times}1)_\#\otimes 1"] & S_\#(A_b) \otimes \mathbb{M}_b \arrow[u, "u^A_b\otimes 1"'] \arrow[uu, bend right=60, "i^A_{b\#}\otimes 1"']
\end{tikzcd}
$$

Proof. The commutativity of the above diagram of solid arrows follows from definition of involved maps (see 3.9.9). Commutativity of the outer square allows us to define $\beta^{\mathcal{U}}_{ab}$ as the restriction map of β^{X}_{ab} to $S(\mathcal{U}\mathcal{J}_{ab})\otimes\mathbb{M}_a$.

3.9.13 Definition. Let ι_a, $a \in \mathrm{ob}\,J$, be the canonical injection of $S_{\#}(\mathcal{U}_a)\otimes\mathbb{M}_a$ into $\sum\limits_{c} S_{\#}(\mathcal{U}_c) \otimes \mathbb{M}_c$. We define chain maps $\alpha^{\mathcal{U}} = \sum\limits_{c} \iota_a\alpha^{\mathcal{U}}_{ab}$ and $\beta^{\mathcal{U}} = \sum\limits_{c} \iota_b\beta^{\mathcal{U}}_{ab}$. Define $\Phi^{\mathcal{U}} = \alpha^{\mathcal{U}} - \beta^{\mathcal{U}}$.

Define $\hat{i}^{\mathcal{U}}_1 = \left(i^{\mathcal{U}\mathcal{J}}_{ab} \otimes 1_{\mathbb{M}_a}\right)_{a,b\,\in\,\mathrm{ob}\,J}$, $\quad \hat{i}^{\mathcal{U}}_2 = \left(i^{\mathcal{U}}_{c} \otimes 1_{\mathbb{M}_c}\right)_{c\,\in\,\mathrm{ob}\,J}$.

Define $\hat{u}^{A}_1 = \left(u^{A}_{ab} \otimes 1_{\mathbb{M}_a}\right)_{a,b\,\in\,\mathrm{ob}\,J}$, $\quad \hat{u}^{A}_2 = \left(u^{A}_{c} \otimes 1_{\mathbb{M}_c}\right)_{c\,\in\,\mathrm{ob}\,J}$.

3.9.14 Lemma.

(1) $\hat{i}^{\mathcal{U}}_1 \circ \hat{u}^{A}_1 = \hat{i}^{A}_1, \quad \hat{i}^{\mathcal{U}}_2 \circ \hat{u}^{A}_2 = \hat{i}^{A}_2.$

(2) The following diagram commutes.

$$
\begin{tikzcd}
\sum\limits_{a,b} S_{\#}(X_b \underline{x} J(a,b))\otimes\mathbb{M}_a \arrow[r, "\Phi^{X}"] & \sum\limits_{c} S_{\#}(X_c)\otimes\mathbb{M}_c \\
\sum\limits_{a,b} S(\mathcal{U}\mathcal{J}_{ab})\otimes\mathbb{M}_a \arrow[r, "\Phi^{\mathcal{U}}"'] \arrow[u, "\hat{i}^{\mathcal{U}}_1"] & \sum\limits_{c} S(\mathcal{U}_c)\otimes\mathbb{M}_c \arrow[u, "\hat{i}^{\mathcal{U}}_2"] \\
\sum\limits_{a,b} S_{\#}(A_b \underline{x} J(a,b))\otimes\mathbb{M}_a \arrow[r, "\Phi^{A}"'] \arrow[u, "\hat{u}^{A}_1"] \arrow[uu, bend left=60, "\hat{i}^{A}_1"] & \sum\limits_{c} S_{\#}(A_c)\otimes\mathbb{M}_c \arrow[u, "\hat{u}^{A}_2"] \arrow[uu, bend right=60, "\hat{i}^{A}_2"']
\end{tikzcd}
$$

Proof. The claim follows by combining 3.9.11 and 3.9.12.

3.9.15 Definition.

(1) Define $C_*^{\mathcal{J}}(\mathcal{U};\ \mathcal{M}) = \operatorname{coker}(\Phi^{\mathcal{U}})$.

(2) Denote by $i^{\mathcal{U}}$ a chain map induced by the pair $(\hat{i}_1^{\mathcal{U}},\ \hat{i}_2^{\mathcal{U}})$.

(3) Denote by $i^A_{\#}$ a chain map induced by the pair $(\hat{i}_1^A,\ \hat{i}_2^A)$.

(4) Denote by u^A a chain map induced by the pair $(\hat{u}_2^A,\ \hat{u}_2^A)$.

3.9.16 Corollary.

(1) $C_*^{\mathcal{J}}(A;\ \mathcal{M}) \xrightarrow{u^A} C_*^{\mathcal{J}}(\mathcal{U};\ \mathcal{M}) \xrightarrow{i^{\mathcal{U}}} C_*^{\mathcal{J}}(X;\ \mathcal{M})$

(2) $i^{\mathcal{U}} \circ u^A = i^A_{\#}$

Proof. The proof follows directly from the definition of maps and 3.9.14

Our next goal is to define a retraction and a subdivision operator determined by the covering.

3.9.17 Lemma. Let $\mathcal{U} = \{A\}$ be a collection of saturated $\mathcal{J}$-subdiagrams of X, such that $Xa = \cup\ \{\operatorname{int}_{X_a}(Aa) \mid A \in \mathcal{U}\}$.

(1) Let $a \in \operatorname{ob} \mathcal{J}$. Then $\mathcal{U}_a = \{A_a \mid A \in \mathcal{U}\}$ is a covering of X_a with property $Xa = \cup\ \{\operatorname{int}_{X_a}(Z)) \mid Z \in \mathcal{U}_a\}$. For any singular simplex σ of Xa, there is an integer $m(\sigma) \geq 0$, such that $sd^{m(\sigma)}\sigma \in S(\mathcal{U}_a)$ $\big($where $sd^{m(\sigma)}\sigma = sd^{m(\sigma)}_{X_a}\sigma\big)$.

(2) Let $a,b \in \operatorname{ob} \mathcal{J}$. Then $\mathcal{W}_{ab} = \{A_a \underline{x} \mathcal{J}(a,b) \mid A \in \mathcal{U}\}$ is a covering of $X_a \underline{x} \mathcal{J}(a,b)$ with property $X_a \underline{x} \mathcal{J}(a,b) = \cup\ \{\operatorname{int}_{X_a x J(a,b)}(A_a \underline{x} \mathcal{J}(a,b)) \mid A \in \mathcal{U}\}$. For any singular simplex (σ,η) of $X_a \underline{x} \mathcal{J}(a,b)$, there is an integer $m(\sigma,\eta) \geq 0$,

such that $sd^{m(\sigma,\eta)}(\sigma,\eta) \in S(\mathcal{U}_{ab})$. Moreover $m((\sigma,\eta)) = m(\sigma)$.

Proof. [Spanier] Lemma 4.4.13 p.178.

3.9.18 Definition.

(1) Let $a \in$ ob J. Define $R_a:S_\#(X_a) \longrightarrow S_\#(X_a)$ by $R_a(\sigma) = \sum_{k=0}^{m(\sigma)-1} D(sd^k(\sigma))$.

Note that R_a is subdivision homotopy associated with the covering $\mathcal{U}_a = \{A_a \mid A \in \mathcal{U}\}$ of X_a. Define $\tau_a:S_\#(X_a) \longrightarrow S(\mathcal{U}_a)$ by

$$\tau_a(\sigma) = \sum_{i=0}^{|\sigma|} \sum_{j=m(\sigma^{(i)})}^{m(\sigma)-1} D(sd^j(\sigma^{(i)})) + sd^{m(\sigma)}(\sigma) \text{ (see [Spanier] p.179).}$$

Recall that the inclusion of $S(\mathcal{U}_a)$ into $S_\#(X_a)$ is denoted by $i_a^{\mathcal{U}}$. From [Spanier], Theorem 4.4.14, we have $\tau_a \circ i_a^{\mathcal{U}} = 1_{S(\mathcal{U}_a)}$ and $R_a:i_a^{\mathcal{U}} \circ \tau_a \simeq 1$. We have a chain map $\tilde{R}_a:\bar{\Delta}(1)\otimes S_\#(X_a) \longrightarrow S_\#(X_a)$ associated with R_a, defined by $\tilde{R}_a(\{0\}\otimes\_) = i_a^{\mathcal{U}} \circ \tau_a$, $\tilde{R}_a(\{1\}\otimes\_) = 1$, and $\tilde{R}_a(\Omega\otimes\_) = R_a$.

(2) Let $a,b \in$ ob J. Define $RJ_{ab}:S_\#(X_b \underline{x} J(a,b)) \longrightarrow S_\#(X_b \underline{x} J(a,b))$ by

$RJ_{ab}(\omega) = \sum_{k=0}^{m(\omega)-1} D(sd^k(\omega))$. Define $\tau_{ab}:S_\#(X_b \underline{x} J(a,b)) \longrightarrow S(\mathcal{U}J_{ab})$ by

$$\tau_{ab}(\omega) = \sum_{i=0}^{|\omega|} \sum_{j=m(\omega^{(i)})}^{m(\omega)-1} D(sd^j(\omega^{(i)})) + sd^{m(\omega)}(\omega) \text{ (see [Spanier] p.179).}$$

Recall that the inclusion of $S(\mathcal{U}J_{ab})$ into $S_\#(X_b \underline{x} J(a,b))$ is denoted by $i_{ab}^{\mathcal{U}J}$. From Spanier, Theorem 4.4.14, we have

$\tau_{ab} \circ i_{ab}^{\mathcal{U}} = 1_{S(\mathcal{U}J_{ab})}$ and $RJ_{ab}:i_{ab}^{\mathcal{U}J} \circ \tau_{ab} \simeq 1$.

We have a chain map $\tilde{R}\mathcal{J}_{ab}:\bar{\Delta}(1)\otimes S_\#(X_b\underline{\times}\mathcal{J}(a,b)) \longrightarrow S_\#(X_b\underline{\times}\mathcal{J}(a,b))$ associated with $R\mathcal{J}_{ab}$, defined by $\tilde{R}\mathcal{J}_{ab}(\{0\}\otimes\_) = i^{\mathcal{W}}_{ab}\circ\tau_{ab}$, $\tilde{R}\mathcal{J}_{ab}(\{1\}\otimes\_) = 1$, and $\tilde{R}\mathcal{J}_{ab}(\Omega\otimes\_) = R\mathcal{J}_{ab}$.

3.9.19 Lemma (for α's). Let $a,b \in$ ob $\mathcal{J}$. The following square commutes:

$$\begin{tikzcd}
S_\#(X_b\underline{\times}\mathcal{J}(a,b))\otimes\mathbb{M}_a \arrow[r, "\alpha^X_{ab}"] & S_\#(X_a)\otimes\mathbb{M}_a \\
\bar{\Delta}(1)\otimes S_\#(X_b\underline{\times}\mathcal{J}(a,b))\otimes\mathbb{M}_a \arrow[u, "\tilde{R}\mathcal{J}_{ab}\otimes 1"] \arrow[r, "1\otimes\alpha^X_{ab}"'] & \bar{\Delta}(1)\otimes S_\#(X_a)\otimes\mathbb{M}_a . \arrow[u, "\tilde{R}_a\otimes 1"']
\end{tikzcd}$$

$\tilde{R}\mathcal{J}_{ab}\otimes 1_{\mathbb{M}_a}$ is a homotopy between $(i^{\mathcal{W}}_{ab}\circ\tau_{ab})\otimes 1_{\mathbb{M}_a}$ and 1. $\tilde{D}_{X_a}$ is a homotopy between $(i^{\mathcal{U}}_a\circ\tau_a)\otimes 1_{\mathbb{M}_a}$ and 1.

Proof. First, we check the commutativity of the following diagram.

$$\begin{tikzcd}
S_\#(X_b\underline{\times}\mathcal{J}(a,b)) \arrow[r, "\tilde{X}_{ab\#}"] & S_\#(X_a) \\
\bar{\Delta}(1)\otimes S_\#(X_b\underline{\times}\mathcal{J}(a,b)) \arrow[u, "\tilde{R}\mathcal{J}_{ab}"] \arrow[r, "1\otimes\tilde{X}_{ab\#}"'] & \bar{\Delta}(1)\otimes S_\#(X_a) \arrow[u, "\tilde{R}_a"']
\end{tikzcd}$$

Let (σ,η) be a singular simplex of $X_b\underline{\times}\mathcal{J}(a,b)$. Note that $m(\sigma,\eta) = m(\sigma)$ and recall that $\tilde{X}_{ab\#}$ commutes with *sd* and D (see 3.9.1). Moreover, since all subdiagrams $A \in \mathcal{U}$ of X are saturated in X, and ω is any singular simplex of

$X_a \underline{x} J(a,b)$, then $\tilde{X}_{ab\#}(\omega) \in \mathrm{image}\left(i_a^{\wedge}(S_\#(A_a))\right)$ if and only if $\omega \in \mathrm{image}\left((i_a^{\wedge} \underline{x} 1)_\#(S_\#(A_a \underline{x} J(a,b)))\right)$. This and the commutativity $sd^j(\tilde{X}_{ab\#}(\omega)) = \tilde{X}_{ab\#}(sd^j(\omega))$, for any j, gives us that $m\left(\tilde{X}_{ab\#}(\omega)\right) = m(\omega)$. Finally $m\left(\tilde{X}_{ab\#}(\sigma,\eta)\right) = m(\sigma,\eta) = m(\sigma)$ for $\omega = (\sigma,\eta)$.

If $\omega = \{0\}\otimes(\sigma,\eta)$, then:

$$\tilde{X}_{ab\#}\left[\tilde{R}J_{ab}(\{0\}\otimes(\sigma,\eta))\right] = \tilde{X}_{ab\#}\left[\tau_{ab}(\sigma,\eta)\right] =$$

$$\tilde{X}_{ab\#}\left[\sum_{i=0}^{|(\sigma,\eta)|}\ \sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} D(sd^j(\sigma^{(i)},\eta^{(i)})) + sd^{m(\sigma,\eta)}(\sigma,\eta)\right] =$$

$$\sum_{i=0}^{|\sigma|}\ \sum_{j=m(\sigma^{(i)})}^{m(\sigma)-1} \tilde{X}_{ab\#}\left[D(sd^j(\sigma^{(i)},\eta^{(i)}))\right] + \tilde{X}_{ab\#}\left[sd^{m(\sigma,\eta)}(\sigma,\eta)\right] =$$

$$\sum_{i=0}^{|\tilde{X}_{ab\#}(\sigma,\eta)|}\ \sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\tilde{X}_{ab\#}(\sigma,\eta))-1} D(sd^j(\tilde{X}_{ab\#}(\sigma,\eta)^{(i)})) + sd^{m(\sigma,\eta)}(\tilde{X}_{ab\#}(\sigma,\eta)) =$$

$$\tau_a(\tilde{X}_{ab\#}(\sigma,\eta))$$

and $\tilde{R}_a\left[1_{\bar{\Delta}(1)}\otimes\tilde{X}_{ab\#}(\{0\}\otimes(\sigma,\eta))\right] = \tilde{R}_a(\{0\}\otimes\tilde{X}_{ab\#}(\sigma,\eta)) = \tau_a(\tilde{X}_{ab\#}(\sigma,\eta))$,

If $\omega = \{1\}\otimes(\sigma,\eta)$, then

$$\tilde{X}_{ab\#}\left[\tilde{R}J_{ab}(\{1\}\otimes(\sigma,\eta))\right] = \tilde{X}_{ab\#}(\sigma,\eta), \text{ and}$$

$$\tilde{R}_a\left[1_{\bar{\Delta}(1)}\otimes\tilde{X}_{ab\#}(\{1\}\otimes(\sigma,\eta))\right] = \tilde{R}_a(\{1\}\otimes\tilde{X}_{ab\#}(\sigma,\eta)) = \tilde{X}_{ab\#}(\sigma,\eta).$$

If $\omega = \Omega\otimes(\sigma,\eta)$, then

$$\tilde{X}_{ab\#}\left[\tilde{R}J_{ab}(\Omega\otimes(\sigma,\eta))\right] = \tilde{X}_{ab\#}(RJ_{ab}(\sigma,\eta)) = \tilde{X}_{ab\#}\left[\sum_{k=0}^{m(\sigma,\eta)-1} D(sd^k(\sigma,\eta))\right] =$$

$$\sum_{k=0}^{m(\tilde{X}_{ab\#}(\sigma,\eta))-1} D(sd^k(\tilde{X}_{ab\#}(\sigma,\eta))) = R_a(\tilde{X}_{ab\#}(\sigma,\eta)) \text{ and}$$

$$\tilde{R}_a\left[1_{\bar{\Delta}(1)}\otimes\tilde{X}_{ab\#}(\Omega\otimes(\sigma,\eta))\right] = \tilde{R}_a(\Omega\otimes\tilde{X}_{ab\#}(\sigma,\eta)) = R_a(\tilde{X}_{ab\#}(\sigma,\eta)).$$

Thus the squares commutes. Tensoring the above diagram with $\mathbb{M}_a$ gives us the claim (see definition of α's in 3.3.4).

3.9.20 Lemma. (for β's). The following diagram commutes.

$$\begin{tikzcd}
S_\#(X_b \underline{x} J(a,b))\otimes\mathbb{M}_a \arrow[r, "\beta^X_{ab}"] & S_\#(X_b)\otimes\mathbb{M}_b \\
\bar{\Delta}(1)\otimes S_\#(X_b \underline{x} J(a,b))\otimes\mathbb{M}_a \arrow[r, "1\otimes\beta^X_{ab}"'] \arrow[u, "\tilde{R}J_{ab}\otimes 1_{\mathbb{M}_a}"] & \bar{\Delta}(1)\otimes S_\#(X_b)\otimes\mathbb{M}_b \arrow[u, "\tilde{R}_b\otimes 1_{\mathbb{M}_b}"']
\end{tikzcd}$$

Proof. For $\{0\}\otimes(\sigma,\eta)\otimes m$, $\beta^X_{ab}\left[\tilde{R}J_{ab}\otimes 1_{\mathbb{M}_a}(\{0\}\otimes(\sigma,\eta)\otimes m)\right] = \beta^X_{ab}(\tau_{ab}(\sigma,\eta)\otimes m) =$

$$\beta^X_{ab}\left(\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} D(sd^j(\sigma^{(i)},\eta^{(i)}))\otimes m + sd^{m(\sigma,\eta)}(\sigma,\eta)\otimes m\right) =$$

$$\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \beta^X_{ab}\left[\tilde{D}_{XbxJ(a,b)}\otimes 1_{\mathbb{M}_a}(\Omega\otimes sd^j(\sigma^{(i)},\eta^{(i)})\otimes m)\right] +$$

$$\beta^X_{ab}\left[(sd\otimes 1)^{m(\sigma,\eta)}((\sigma,\eta)\otimes m)\right].$$

In view of Lemmas 3.9.3 and 3.9.6, the above sum becomes

$$\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \tilde{D}_{Xb}\otimes 1_{\mathbb{M}_b}\left[1_{\bar{\Delta}(1)}\otimes\beta^X_{ab}(\Omega\otimes sd^j(\sigma^{(i)},\eta^{(i)})\otimes m)\right] +$$

$$(sd\otimes 1_{\mathbb{M}_b})^{m(\sigma,\eta)}\left[\beta^X_{ab}((\sigma,\eta)\otimes m)\right] =$$

$$\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \tilde{D}_{Xb}\otimes 1_{\mathbb{M}b}\left[\Omega\otimes\beta^{X}_{ab}(sd^{j}(\sigma^{(i)},\eta^{(i)})\otimes m)\right] +$$

$$sd^{m(\sigma,\eta)}\otimes 1_{\mathbb{M}b}(\sigma\otimes\check{\mathbb{M}}(\eta)(m)) =$$

$$\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \tilde{D}_{Xb}\otimes 1_{\mathbb{M}b}\left[\Omega\otimes\beta^{X}_{ab}((sd\otimes 1_{\mathbb{M}a})^{j}((\sigma^{(i)},\eta^{(i)})\otimes m))\right] +$$

$$sd^{m(\sigma,\eta)}(\sigma)\otimes\check{\mathbb{M}}(\eta)(m) =$$

$$\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \tilde{D}_{Xb}\otimes 1_{\mathbb{M}b}\left[\Omega\otimes(sd\otimes 1_{\mathbb{M}b})^{j}(\beta^{X}_{ab}((\sigma^{(i)},\eta^{(i)})\otimes m))\right] +$$

$$sd^{m(\sigma,\eta)}(\sigma)\otimes\check{\mathbb{M}}(\eta)(m) =$$

$$\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \tilde{D}_{Xb}\otimes 1_{\mathbb{M}b}\left[\Omega\otimes(sd^{j}\otimes 1_{\mathbb{M}b}(\sigma^{(i)}\otimes\check{\mathbb{M}}(\eta^{(i)})(m)))\right] +$$

$$sd^{m(\sigma,\eta)}(\sigma)\otimes\check{\mathbb{M}}(\eta)(m) =$$

$$\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \tilde{D}_{Xb}\otimes 1_{\mathbb{M}b}\left[\Omega\otimes sd^{j}(\sigma^{(i)})\otimes\check{\mathbb{M}}(\eta^{(i)})(m)\right] + sd^{m(\sigma,\eta)}(\sigma)\otimes\check{\mathbb{M}}(\eta)(m) =$$

$$\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} D_{Xb}(sd^{j}(\sigma^{(i)}))\otimes\check{\mathbb{M}}(\eta^{(i)})(m) + sd^{m(\sigma,\eta)}(\sigma)\otimes\check{\mathbb{M}}(\eta)(m) =$$

On the other hand, $\tilde{R}_{b}\otimes 1_{\mathbb{M}b}\left[1_{\bar{\Delta}(1)}\otimes\beta^{X}_{ab}(\{0\}\otimes(\sigma,\eta)\otimes m)\right] =$ $\tilde{R}_{b}\otimes 1_{\mathbb{M}b}(\{0\}\otimes\sigma\otimes\check{\mathbb{M}}(\eta)(m)) = \tau_{b}(\sigma)\otimes\check{\mathbb{M}}(\eta)(m) = \displaystyle\sum_{i=0}^{|\sigma|}\sum_{j=m(\sigma^{(i)})}^{m(\sigma)-1} D(sd^{j}(\sigma^{(i)}))\otimes\check{\mathbb{M}}(\eta)(m) +$ $sd^{m(\sigma)}(\sigma)\otimes\check{\mathbb{M}}(\eta)(m)$. Since $\check{\mathbb{M}}(\eta^{(i)}) = \check{\mathbb{M}}(\eta^{(i)})$, we have

$$\beta^{X}_{ab}\left[\tilde{R}J_{ab}\otimes 1_{\mathbb{M}a}(\{0\}\otimes(\sigma,\eta)\otimes m)\right] = \tilde{R}_{b}\otimes 1_{\mathbb{M}b}\left[1_{\bar{\Delta}(1)}\otimes\beta^{X}_{ab}(\{0\}\otimes(\sigma,\eta)\otimes m)\right].$$

For $\{0\}\otimes(\sigma,\eta)\otimes m$, we have

$$\beta^X_{ab}\left(\tilde{R}J_{ab}\otimes 1_{M_a}(\{1\}\otimes(\sigma,\eta)\otimes m)\right) = \beta^X_{ab}((\sigma,\eta)\otimes m) = \sigma\otimes\check{M}(\eta)(m), \text{ and}$$

$$\tilde{R}_b\otimes 1_{M_b}\left(1_{\bar{\Delta}(1)}\otimes\beta^X_{ab}(\{1\}\otimes(\sigma,\eta)\otimes m)\right) = \tilde{R}_b\otimes 1_{M_b}\left(\{1\}\otimes\beta^X_{ab}((\sigma,\eta)\otimes m)\right) =$$

$$\tilde{R}_b\otimes 1_{M_b}\left(\{1\}\otimes\sigma\otimes\check{M}(\eta)(m)\right) = \sigma\otimes\check{M}(\eta)(m), \text{ and thus}$$

$$\beta^X_{ab}\left(\tilde{R}J_{ab}\otimes 1_{M_a}(\{1\}\otimes(\sigma,\eta)\otimes m)\right) = \tilde{R}_b\otimes 1_{M_b}\left(1_{\bar{\Delta}(1)}\otimes\beta^X_{ab}(\{1\}\otimes(\sigma,\eta)\otimes m)\right).$$

For $\Omega\otimes(\sigma,\eta)\otimes m$, we have

$$\beta^X_{ab}\left(\tilde{R}J_{ab}\otimes 1_{M_a}(\Omega\otimes(\sigma,\eta)\otimes m)\right) = \beta^X_{ab}\left(RJ_{ab}(\sigma,\eta)\otimes m\right) =$$

$$\beta^X_{ab}\left(\sum_{k=0}^{m(\sigma,\eta)-1} D(sd^k(\sigma,\eta))\otimes m\right) = \sum_{k=0}^{m(\sigma,\eta)-1} \beta^X_{ab}\left(\tilde{D}\otimes 1_{M_a}(\Omega\otimes sd^k(\sigma,\eta)\otimes m)\right) =$$

$$\sum_{k=0}^{m(\sigma,\eta)-1} \tilde{D}\otimes 1_{M_b}\left(1_{\bar{\Delta}(1)}\otimes\beta^X_{ab}(\Omega\otimes sd^k(\sigma,\eta)\otimes m)\right) = \sum_{k=0}^{m(\sigma,\eta)-1} \tilde{D}\otimes 1_{M_b}\left(\Omega\otimes\beta^X_{ab}(sd^k(\sigma,\eta)\otimes m)\right) =$$

$$\sum_{k=0}^{m(\sigma,\eta)-1} \tilde{D}\otimes 1_{M_b}\left(\Omega\otimes\beta^X_{ab}((sd\otimes 1_{M_a})^k((\sigma,\eta)\otimes m))\right) =$$

$$\sum_{k=0}^{m(\sigma,\eta)-11} \tilde{D}\otimes 1_{M_b}\left(\Omega\otimes(sd\otimes 1_{M_b})^k(\beta^X_{ab}((\sigma,\eta)\otimes m))\right) =$$

$$\sum_{k=0}^{m(\sigma,\eta)-11} \tilde{D}\otimes 1_{M_b}\left(\Omega\otimes(sd^k\otimes 1_{M_b})(\sigma\otimes\check{M}(\eta)(m))\right) = \sum_{k=0}^{m(\sigma,\eta)-11} \tilde{D}\otimes 1_{M_b}\left(\Omega\otimes sd^k(\sigma)\otimes\check{M}(\eta)(m))\right) =$$

$$\sum_{k=0}^{m(\sigma,\eta)-1} D(sd^k(\sigma))\otimes\check{M}(\eta)(m).$$

On the other hand,

$$\tilde{R}_b\otimes 1_{M_b}\left(1\otimes\beta^X_{ab}(\Omega\otimes(\sigma,\eta)\otimes m)\right) = \tilde{R}_b\otimes 1_{M_b}\left(\Omega\otimes\beta^X_{ab}((\sigma,\eta)\otimes m)\right) = \tilde{R}_b\otimes 1_{M_b}(\Omega\otimes\sigma\otimes\check{M}(\eta)(m)) =$$

$$R_b(\sigma)\otimes\check{M}(\eta)(m) = \sum_{k=0}^{m(\sigma)-1} D(sd^k(\sigma))\otimes\check{M}(\eta)(m).$$

Therefore, $\beta^X_{ab}\left[\tilde{R}J_{ab}\otimes 1_{M_a}(\Omega\otimes(\sigma,\eta)\otimes m)\right] = \tilde{R}_b\otimes 1_{M_b}\left[1\otimes\beta^X_{ab}(\Omega\otimes(\sigma,\eta)\otimes m)\right]$.

3.9.21 Lemma. $\tilde{R}$ induces the following commutative square of chain maps

$$\begin{array}{ccc}
\sum\limits_{a,b} S_\#(X_b \underline{x} J(a,b))\otimes M_a & \xrightarrow{\Phi^X} & \sum\limits_{c} S_\#(X_c)\otimes M_c \\
{\scriptstyle (\tilde{R}J_{ab}\otimes 1_{M_a})}\uparrow & & \uparrow{\scriptstyle (\tilde{R}_c\otimes 1_{M_c})} \\
\bar{\Delta}(1)\otimes \sum\limits_{a,b} S_\#(X_b \underline{x} J(a,b))\otimes M_a & \xrightarrow[1\otimes\Phi^X]{} & \bar{\Delta}(1)\otimes \sum\limits_{c} S_\#(X_c)\otimes M_c
\end{array}$$

Proof. Combine 3.9.19 and 3.9.20 and the definition of Φ.

3.9.22 Definition. Define

$$\bar{\tau}_1 = \left(\tau_{ab}\otimes 1_{M_a}\right)_{a,b\,\in\,\mathrm{ob}\,J} : \sum_{a,b} S_\#(X_b \underline{x} J(a,b))\otimes M_a \longrightarrow \sum_{a,b} S(\mathcal{U}J_{ab})\otimes M_a$$

$$\bar{\tau}_2 = \left(\tau_c\otimes 1_{M_c}\right)_{c\,\in\,\mathrm{ob}\,J} : \sum_{c} S_\#(X_c)\otimes M_c \longrightarrow \sum_{c} S(\mathcal{U}_c)\otimes M_c$$

Denote by $\hat{R}_1 = \left(\tilde{R}J_{ab}\otimes 1_{M_a}\right)_{a,b\,\in\,\mathrm{ob}\,J}$ and $\hat{R}_2 = \left(\tilde{R}_a\otimes 1_{M_c}\right)_{c\,\in\,\mathrm{ob}\,J}$, the two vertical maps in Lemma 3.9.21.

3.9.23 Proposition.

(1) $\hat{R}_1$ is a homotopy between $\hat{i}^{\mathcal{U}}_1 \circ \bar{\tau}_1$ and 1, moreover $\bar{\tau}_1 \circ \hat{i}^{\mathcal{U}}_1 = 1$.

(2) $\hat{R}_2$ is a homotopy between $\hat{i}_2^{\mathcal{U}} \circ \bar{\tau}_2$ and 1, moreover $\bar{\tau}_2 \circ \hat{i}_2^{\mathcal{U}} = 1$.

(3) τ induces the following commutative square of chain maps:

$$\begin{array}{ccc}
\sum\limits_{a,b} S(\mathcal{UJ}_{ab})\otimes\mathbb{M}_a & \xrightarrow{\Phi^{\mathcal{U}}} & \sum\limits_{c} S(\mathcal{U}_c)\otimes\mathbb{M}_c \\
\bar{\tau}_1 = (\tau_{ab}\otimes 1_{\mathbb{M}_a}) \Big\uparrow & & \Big\uparrow (\tau_c\otimes 1_{\mathbb{M}_c}) = \bar{\tau}_2 \\
\sum\limits_{a,b} S_{\#}(X_b \underline{x} \mathcal{J}(a,b))\otimes\mathbb{M}_a & \xrightarrow[\Phi^{X}]{} & \sum\limits_{c} S_{\#}(X_c)\otimes\mathbb{M}_c
\end{array}$$

Proof. Note that $\hat{R}_1 \circ \hat{i}_0 = \left(\tilde{R}\mathcal{J}_{ab}(\{0\}\otimes\_) \otimes 1_{\mathbb{M}_a}\right)_{a,b\, \in\, \text{ob}\, \mathcal{J}} =$
$\left((i_{ab}^{\mathcal{UJ}} \circ \tau_{ab}) \otimes 1_{\mathbb{M}_a}\right)_{a,b\, \in\, \text{ob}\, \mathcal{J}} = \left(i_{ab}^{\mathcal{UJ}} \otimes 1_{\mathbb{M}_a}\right)_{a,b\, \in\, \text{ob}\, \mathcal{J}} \circ \left(\tau_{ab} \otimes 1_{\mathbb{M}_a}\right)_{a,b\, \in\, \text{ob}\, \mathcal{J}}$
$= \hat{i}_1^{\mathcal{U}} \circ \bar{\tau}_1$. Also $\hat{R}_1 \circ \hat{i}_2 = \left(\tilde{R}\mathcal{J}_{ab}(\{1\}\otimes\_)\otimes 1_{\mathbb{M}_a}\right)_{a,b\, \in\, \text{ob}\, \mathcal{J}} = \left(1\otimes 1_{\mathbb{M}_a}\right)_{a,b\, \in\, \text{ob}\, \mathcal{J}} = 1$.

Similarly, $\hat{R}_2 \circ \hat{i}_1 = \left(\tilde{R}_c(\{0\}\otimes\_) \otimes 1_{\mathbb{M}_c}\right)_{c\, \in\, \text{ob}\, \mathcal{J}} = \left((i_c^{\mathcal{U}} \circ \tau_c) \otimes 1_{\mathbb{M}_c}\right)_{c\, \in\, \text{ob}\, \mathcal{J}} =$
$\left(i_c^{\mathcal{U}} \otimes 1_{\mathbb{M}_c}\right)_{c\, \in\, \text{ob}\, \mathcal{J}} \circ \left(\tau_c \otimes 1_{\mathbb{M}_c}\right)_{c\, \in\, \text{ob}\, \mathcal{J}} = \hat{i}_2^{\mathcal{U}} \circ \bar{\tau}_2$ and
$\hat{R}_1 \circ \hat{i}_1 = \left(\tilde{R}_c(\{1\}\otimes\_) \otimes 1_{\mathbb{M}_c}\right)_{c\, \in\, \text{ob}\, \mathcal{J}} = \left(1 \otimes 1_{\mathbb{M}_c}\right)_{c\, \in\, \text{ob}\, \mathcal{J}} = 1$.
Commutativity of 3.9.20 proves (3).

3.9.24 Remark. We summarize commutativity of the maps in Proposition 3.9.20 by the following commutative diagram:

$$\begin{tikzcd}
\sum\limits_{a,b} S_\#(X_b \underline{x} J(a,b))\otimes \mathbb{M}_a \arrow[r, "\Phi^X"] & \sum\limits_{c} S_\#(X_c)\otimes \mathbb{M}_c \\
\bar{\Delta}(1)\otimes \sum\limits_{a,b} S_\#(X_b \underline{x} J(a,b))\otimes \mathbb{M}_a \arrow[r, "1\otimes\Phi^X"'] \arrow[u, "\hat{R}_1 = (\tilde{R}J_{ab}\otimes 1_{\mathbb{M}_a})"] & \bar{\Delta}(1)\otimes \sum\limits_{c} S_\#(X_c)\otimes \mathbb{M}_c \arrow[u, "(\tilde{R}_c\otimes 1_{\mathbb{M}_c}) = \hat{R}_2"'] \\
\sum\limits_{a,b} S_\#(X_b \underline{x} J(a,b))\otimes \mathbb{M}_a \arrow[r, "\Phi"] \arrow[u, "\hat{i}_1"] & \sum\limits_{c} S_\#(X_c)\otimes \mathbb{M}_c \arrow[u, "\hat{i}_2"']
\end{tikzcd}$$

The commutative diagram in 3.9.24 gives us the following proposition.

3.9.25 Proposition. $\tilde{R}$ induces a homotopy $\tilde{R}:\bar{\Delta}(1)\otimes C^{J}_{\bullet}(X;\ \mathbb{M}) \longrightarrow C^{J}_{\bullet}(X;\ \mathbb{M})$ between $i^{\mathcal{U}}\circ\bar{\tau}$ and 1, and $\bar{\tau}\circ i^{\mathcal{U}} = 1$.

Clearly 3.9.25 yields that $i^{\mathcal{U}}:C^{J}_{\bullet}(\mathcal{U};\ \mathbb{M}) \longrightarrow C^{J}_{\bullet}(X;\ \mathbb{M})$ is a chain homotopy equivalence.

3.9.26 Theorem (GENERALIZED EXCISION). $i^{\mathcal{U}}_{\bullet}:H^{J}_{\bullet}(\mathcal{U};\ \mathbb{M}) \longrightarrow H^{J}_{\bullet}(X;\ \mathbb{M})$ is an isomorphism.

The above theorem is stated for a single diagram X, but it is obvious that we can perform the previous constructions for a saturated pair (X, B) and a covering $\mathcal{U}$ of X with the additional assumption that each intersection space $A_a \cap B_a$ is a k-space, for $a \in$ ob J. Therefore we have a version of the generalized excision theorem for J-pairs. We denote by $i^{\mathcal{U}(X,B)}:C^{J}_{\bullet}(\mathcal{U},\mathcal{U}\cap B;\mathbb{M})$

$\longrightarrow C^J_*(X,B;\mathbb{M})$ the induced inclusion morphism between the complex of $\mathcal{U}$–small singular J–chains and $C^J_*(X,B;\mathbb{M})$ (compare with $i^{\mathcal{U}}$ in 3.9.15(2)).

3.9.27 Theorem (GENERALIZED EXCISION). $i^{\mathcal{U}(X,B)}{}_*: H^J_*(\mathcal{U},\mathcal{U}\cap B;\ \mathbb{M}) \longrightarrow H^J_*(X,B;\mathbb{M})$ is an isomorphism.

3.9.28 Let (X, B) be a saturated J–pair and U be an open saturated subdiagram of X such that $\bar{U} \subseteq B$. Let $\mathcal{V} = \{B, X\text{-}U\}$. First, $\mathcal{V}$ is a covering of X since $X_a = \mathrm{int}(B_a) \cup (X_a\text{-}\mathrm{int}(B)) \subseteq \mathrm{int}(B_a) \cup (X_a\text{-}\bar{U}_a) \subseteq \mathrm{int}(B_a) \cup \mathrm{int}(X_a\text{-}U_a)$, $a \in \mathrm{ob}\ J$. Second, we check that the restriction of $\mathcal{V}$ to B is satisfactory. For $a \in \mathrm{ob}\ J$, both $B_a \cap B_a = B_a$ and $B_a \cap (X_a\text{-}U_a) = X_a\text{-}U_a$ are k–spaces. Our next goal is to describe $C^J_*(\mathcal{V},\mathcal{V}\cap B;\ \mathbb{M})$ in detail.

3.9.29 Lemma.

(1) There exists an natural isomorphism $C^J_*(\mathcal{V},\mathcal{V}\cap B;\ \mathbb{M}) \cong C^J_*(X\text{-}U,B\text{-}U;\ \mathbb{M})$.

(2) $i^{\mathcal{V}(X,B)}: C^J_*(\mathcal{V},\mathcal{V}\cap B;\ \mathbb{M}) \longrightarrow C^J_*(X,B;\ \mathbb{M})$ (see 3.9.26) under the isomorphism in (1) can be identified as a morphism induced by the inclusion $u:(X\text{-}U,B\text{-}U) \subseteq (X,\ B)$.

Proof. First note that trivially $(B,\ B\cap B) = (B,\ B\cap B)$ and $(X\text{-}U,\ (X\text{-}U)\cap B) = (X\text{-}U,\ B\text{-}U)$. Then $S_\#\big((B_b,\ B_b\cap B_b)\underline{x}J(a,b)\big) = 0$ and $S_\#\big((X_b\text{-}U_b,\ (X_b\text{-}U_b)\cap B_b)\underline{x}J(a,b)\big) = S_\#\big((X_b\text{-}U_b,\ B_b\text{-}U_b)\underline{x}J(a,b)\big)$, and thus canonically $S\mathcal{V}(X,B)J_{ab} \cong S_\#\big((X_b\text{-}U_b,\ B_b\text{-}U_b)\underline{x}J(a,b)\big)$, for $a,b \in \mathrm{ob}\ J$ (compare 3.9.9(2)).

Similarly, $S_\#\big((B_c,\ B_c\cap B_c)\big) = 0$ and $S_\#\big((X_c\text{-}U_c,\ (X_c\text{-}U_c)\cap B_c)\big) = S_\#(X_c\text{-}U_c,\ B_c\text{-}U_c)$, and then canonically $S\mathcal{V}(X,B)_c \cong S_\#(X_c\text{-}U_c,\ B_c\text{-}U_c)$, for $c \in \mathrm{ob}\ J$ (compare 3.9.9(1)). Therefore $\Phi^{\mathcal{U}(X,B)} \cong \Phi^{X,B}$ and we obtain the natural isomorphism of the cokernels: $C^J_*(\mathcal{V},\mathcal{V}\cap B;\ \mathbb{M}) \cong C^J_*(X\text{-}U,B\text{-}U;\ \mathbb{M})$.

3.9.30 Theorem (EXCISION). Let (X,A) be a saturated pair. Let U be an open saturated subdiagram of **X** whose closure $\bar{U}$ is contained in the interior of A. Then, the inclusion morphism $u:(X\text{-}U,A\text{-}U) \longrightarrow (X,A)$ is admissible and it induces an isomorphism $u_*:H_*(X\text{-}U,A\text{-}U) \cong H_*(X,A)$.

Proof. From theorem (3.9.27) for the covering $\mathcal{V} = \{B,\ X\text{-}U\}$, the canonical inclusion $i^{\mathcal{V}}:C^J_*(\mathcal{V},\mathcal{V}\cap B;\ \mathbb{M}) \subseteq C^J_*(X,B;\ \mathbb{M})$ induces isomorphism in homology. An application of Lemma 3.9.29 yields the claim.

3.10 Excision Axiom for J–Cohomology

3.10.1 Let $f:Z \longrightarrow W$ be a map of topological spaces and K be a R-module. The functoriality of the subdivision functor sd_Z in Z, as described in 3.9.1, induces the following commutative diagram:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(S_\#(W),\ K) & \xrightarrow{f^\#} & \underline{\mathrm{Hom}}(S_\#(Z),\ K) \\ {\scriptstyle sd^W}\uparrow & & \uparrow{\scriptstyle sd^Z} \\ \underline{\mathrm{Hom}}(S_\#(W),\ K) & \xrightarrow[f^\#]{} & \underline{\mathrm{Hom}}(S_\#(Z),\ K) \end{array}$$

where sd^Z, induced by taking hom of sd_Z with K, is given explicitly by $sd^Z(f)(\sigma) = f(sd_Z(\sigma))$. Similarly, the functoriality of the chain homotopy D_Z ($D_Z:sd_Z \simeq 1$; degree +1) in Z, as described in 3.9.1, induces a cochain homotopy $D^Z:\underline{\mathrm{Hom}}(S_\#(Z),\ K) \longrightarrow \underline{\mathrm{Hom}}(S_\#(Z),\ K)$ degree -1 and the following commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(S_\#(W), K) \arrow[r, "f^\#"] & \underline{\mathrm{Hom}}(S_\#(Z), K) \\
\underline{\mathrm{Hom}}(S_\#(W), K) \arrow[u, "D^W"] \arrow[r, "f^\#"'] & \underline{\mathrm{Hom}}(S_\#(Z), K), \arrow[u, "D^Z"']
\end{tikzcd}$$

where $D^Z(f)(\sigma) = f(D_Z(\sigma))$. The cochain homotopy D^Z induces a cochain map $\tilde{D}^Z:\bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(Z), K) \longrightarrow \underline{\mathrm{Hom}}(S_\#(Z), K)$ by $\tilde{D}^Z(\{0\}\otimes f) = sd^Z(f)$, $\tilde{D}^Z(\{1\}\otimes f)=f$, and $\tilde{D}^Z(\Omega\otimes f) = D^Z(f)$. Clearly D^Z is functorial in Z, that is, the following square commutes:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(S_\#(W), K) \arrow[r, "f^\#"] & \underline{\mathrm{Hom}}(S_\#(Z), K) \\
\bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(W), K) \arrow[u, "\tilde{D}^W"] \arrow[r, "f^\#"'] & \bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(Z), K). \arrow[u, "\tilde{D}^Z"']
\end{tikzcd}$$

For the remainder of this section $X \in$ ob J-CGV and κ is a contravariant coefficient system on J.

3.10.2 Lemma (for ϕ's). Let $a,b \in$ ob J. The following diagram commutes.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(S_{\#}(X_a), \mathbb{K}_a) \arrow[r, "\phi^X_{ab}"] & \underline{\mathrm{Hom}}(S_{\#}(X_b \underline{\times} J(a,b)), \mathbb{K}_a) \\
\underline{\mathrm{Hom}}(S_{\#}(X_a), \mathbb{K}_a) \arrow[r, "\phi^X_{ab}"] \arrow[u, "sd^{X_a}"] & \underline{\mathrm{Hom}}(S_{\#}(X_b \underline{\times} J(a,b)), \mathbb{K}_a) \arrow[u, "sd^{X_b \times J(a,b)}"']
\end{tikzcd}$$

Proof. Take hom of the diagram in the proof of 3.9.2 with $\mathbb{K}_a$ and observe that $\tilde{X}_{ab}{}^{\#} = \phi^X_{ab}$.

3.10.3 Lemma (for ψ's). Let $a,b \in$ ob J. The following diagram commutes.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(S_{\#}(X_b), \mathbb{K}_b) \arrow[r, "\psi^X_{ab}"] & \underline{\mathrm{Hom}}(S_{\#}(X_b \underline{\times} J(a,b)), \mathbb{K}_a) \\
\underline{\mathrm{Hom}}(S_{\#}(X_b), \mathbb{K}_b) \arrow[r, "\psi^X_{ab}"] \arrow[u, "sd^{X_b}"] & \underline{\mathrm{Hom}}(S_{\#}(X_b \underline{\times} J(a,b)), \mathbb{K}_a) \arrow[u, "sd^{X_b \times J(a,b)}"']
\end{tikzcd}$$

Proof. By the direct calculation we find $\psi^X_{ab}(sd^{X_a}(f))_{p,q}(\sigma,\eta) = \check{\mathbb{K}}(\eta)(sd^{X_a}(f)_{p,q}(\sigma)) = \check{\mathbb{K}}(\eta)(f_{p,q}(sd(\sigma))) = \check{\mathbb{K}}(\eta)(f_{p,q}(\sum_{k=0}^{N(n)} z_k \quad \sigma\circ\omega_k)) = \sum_{k=0}^{N(n)} z_k$ $\check{\mathbb{K}}(\eta)(f_{p,q}(\sigma\circ\omega_k))$. Also $sd^{X_b \times J(a,b)}(\psi^X_{ab}(f))_{p,q}(\sigma,\eta) =$ $\psi^X_{ab}(f)_{p,q}(sd_{X_b \times J(a,b)}(\sigma,\eta)) = \psi^X_{ab}(f)_{p,q}(\sum_{k=0}^{N(n)} z_k(\sigma\circ\omega_k,\eta\circ\omega_k)) =$ $\sum_{k=0}^{N(n)} z_k \psi^X_{ab}(f)_{p,q}(\sigma\circ\omega_k,\eta\circ\omega_k) = \sum_{k=0}^{N(n)} z_k \quad \check{\mathbb{K}}(\eta\circ\omega_k)(f_{p,q}(\sigma\circ\omega_k))$. Since $\check{\mathbb{K}}(\eta\circ\omega_k) =$ $\check{\mathbb{K}}(\eta)$, $\psi^X_{ab}(sd^{X_a}(f))_{p,q}(\sigma,\eta) = sd^{X_b \times J(a,b)}(\psi^X_{ab}(f))_{p,q}(\sigma,\eta)$ as required.

3.10.4 Proposition. The functor *sd* induces the following commutative square of cochain maps

$$\begin{array}{ccc}
\prod\limits_{a} \underline{\mathrm{Hom}}(S_{\#}(X_a), K_a) & \xrightarrow{\Psi^X} & \prod\limits_{b,c} \underline{\mathrm{Hom}}(S_{\#}(X_c \times \mathcal{J}(b,c)), K_b) \\
\Big\uparrow (sd^{X_a}) & & \Big\uparrow (sd^{X_c \times J(b,c)}) \\
\prod\limits_{a} \underline{\mathrm{Hom}}(S_{\#}(X_a), K_a) & \xrightarrow[\Psi^X]{} & \prod\limits_{b,c} \underline{\mathrm{Hom}}(S_{\#}(X_c \times \mathcal{J}(b,c)), K_b).
\end{array}$$

Proof. The claim follows by combining 3.10.2 and 3.10.3 and the definition of Ψ.

3.10.5 Definition. Denote $\check{sd}_1 = \left(sd^{X_a}\right)_{a \in \mathrm{ob}\, \mathcal{J}}$, $\check{sd}_2 = \left(sd^{X_a \times J(a,b)}\right)_{a,b \in \mathrm{ob}\, \mathcal{J}}$.

The commutative diagram in 3.10.4 gives us the following corollary.

Corollary. The functor *sd* induces a cochain map $\check{sd}: C^*_{\mathcal{J}}(X; K) \longrightarrow C^*_{\mathcal{J}}(X; K)$.

Our next goal is to establish that D induces a homotopy on $C^*_J(X; K)$ between $\check{sd}$ and 1.

3.10.6 Lemma (for ϕ's). Let a,b $\in$ ob $\mathcal{J}$. The following diagram commutes:

$$\begin{array}{ccc}
\underline{\mathrm{Hom}}(S_\#(X_a),\, \mathbb{K}_a) & \xrightarrow{\phi^X_{ab}} & \underline{\mathrm{Hom}}(S_\#(X_b \underline{x} J(a,b)),\, \mathbb{K}_a) \\
\tilde{D}^{X_a}\Big\uparrow & & \Big\uparrow \tilde{D}^{X_b x J(a,b)} \\
\bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(X_a),\, \mathbb{K}_a) & \xrightarrow[1\otimes\phi^X_{ab}]{} & \bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(X_b \underline{x} J(a,b)),\, \mathbb{K}_a).
\end{array}$$

Proof. We shall show that $\phi^X_{ab}(\tilde{D}^{X_a}(\omega\otimes f)) = \tilde{D}^{X_a x J(a,b)}(1\otimes\phi^X_{ab}(\omega\otimes f))$ for ω generators of $\bar{\Delta}^{op}(1)$.

The case $\omega = \{0\}$ follows easily from 3.10.2 since $\tilde{D}^Z(\{0\}\otimes f) = sd^Z(f)$.

The case $\omega = \{1\}$ turns into an identity since $\tilde{D}^Z(\{1\}\otimes f) = f$.

Let $\omega = \Omega$. Since $\tilde{D}^Z(\Omega\otimes f) = D^Z(f)$ and $\phi^X_{ab} = \tilde{X}_{ab}{}^\#$, the equality follows from the functoriality of $\tilde{D}$ as discussed in 3.10.1.

3.10.7 Lemma (for ψ's). Let a,b $\in$ ob J. The following diagram commutes:

$$\begin{array}{ccc}
\underline{\mathrm{Hom}}(S_\#(X_b),\, \mathbb{K}_b) & \xrightarrow{\psi^X_{ab}} & \underline{\mathrm{Hom}}(S_\#(X_b \underline{x} J(a,b)),\, \mathbb{K}_a) \\
\tilde{D}^{X_b}\Big\uparrow & & \Big\uparrow \tilde{D}^{X_b x J(a,b)} \\
\bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(X_b),\, \mathbb{K}_b) & \xrightarrow[1\otimes\psi^X_{ab}]{} & \bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(X_b \underline{x} J(a,b)),\, \mathbb{K}_a).
\end{array}$$

Proof. We shall show that $\psi^X_{ab}(\tilde{D}^{X_a}(\omega\otimes f)) = \tilde{D}^{X_a x J(a,b)}(1\otimes\psi^X_{ab}(\omega\otimes f))$ for ω generators of $\bar{\Delta}^{op}(1)$.

The case $\omega = \{0\}$ follows easily from 3.10.3 since $\tilde{D}^Z(\{0\}\otimes f) = sd^Z(f)$.

The case $\omega = \{1\}$ turns into an identity since $\tilde{D}^Z(\{1\}\otimes f) = f$.

Let $\omega = \Omega$. Then $\psi^X_{ab}(\tilde{D}^{X_a}(\Omega\otimes f))_{p,q}(\sigma,\eta) = \psi^X_{ab}(D^{X_a}(f))_{p,q}(\sigma,\eta) =$ $\psi^X_{ab}(D^{X_a}(f))_{p,q}(\sigma,\eta) = \check{\kappa}(\eta)(D^{X_a}(f)_{p,q}(\sigma)) = \check{\kappa}(\eta)(f_{p,q}(D_{x_a}(\sigma))) =$

$$\check{\kappa}(\eta)(f_{p,q}(\sum_{k=0}^{M(n)} \bar{z}_k\ \sigma\circ\bar{\omega}_k)) = \sum_{k=0}^{M(n)} \bar{z}_k\ \check{\kappa}(\eta)(f_{p,q}(\sigma\circ\bar{\omega}_k)).$$

On the other hand $\tilde{D}^{X_a xJ(a,b)}(1\otimes\psi^X_{ab}(\Omega\otimes f))_{p,q}(\sigma,\eta) =$ $\tilde{D}^{X_a xJ(a,b)}(\Omega\otimes\psi^X_{ab}(f))_{p,q}(\sigma,\eta) = D^{X_a xJ(a,b)}(\psi^X_{ab}(f))_{p,q}(\sigma,\eta) =$

$$\psi^X_{ab}(f)_{p,q}(D_{XbxJ(a,b)}(\sigma,\eta)) = \psi^X_{ab}(f)_{p,q}(\sum_{k=0}^{M(n)} \bar{z}_k\ (\sigma\circ\bar{\omega}_k,\eta\circ\bar{\omega}_k)) = \sum_{k=0}^{M(n)} \bar{z}_k$$

$$\psi^X_{ab}(f)_{p,q}((\sigma\circ\bar{\omega}_k,\eta\circ\bar{\omega}_k)) = \sum_{k=0}^{M(n)} \bar{z}_k\ \check{\kappa}(\eta\circ\bar{\omega}_k)(f_{p,q}(\sigma\circ\bar{\omega}_k)).$$

Since $\check{\kappa}(\eta\circ\omega_k) = \check{\kappa}(\eta)$, $\tilde{D}^{X_a xJ(a,b)}(1\otimes\psi^X_{ab}(\Omega\otimes f))_{p,q}(\sigma,\eta) =$ $\psi^X_{ab}(f)_{p,q}(D_{XbxJ(a,b)}(\sigma,\eta))$ as required.

3.10.8 Proposition. The functor $\tilde{D}$ induces the following commutative square of cochain maps:

$$\begin{tikzcd}
\prod_a \underline{\mathrm{Hom}}(S_\#(X_a), \kappa_a) \arrow[r, "\Psi^X"] & \prod_{b,c} \underline{\mathrm{Hom}}(S_\#(X_c \underline{x} J(b,c)), \kappa_b) \\
\bar{\Delta}^{op}(1)\otimes\left(\prod_a \underline{\mathrm{Hom}}(S_\#(X_a), \kappa_a)\right) \arrow[r, "1\otimes\Psi^X"'] \arrow[u, "(\tilde{D}^{X_a})"] & \bar{\Delta}^{op}(1)\otimes\left(\prod_{b,c} \underline{\mathrm{Hom}}(S_\#(X_c \underline{x} J(b,c)), \kappa_b)\right) \arrow[u, "(D^{X_c x J(b,c)})"']
\end{tikzcd}$$

Proof. The claim follows by combining 3.10.6 and 3.10.7 and the definition of Ψ.

3.10.9 Definition. Denote

$$\check{D}_1 = \left(\tilde{D}^{X_a}\right)_{a \in \text{ob } \mathcal{J}}, \quad \check{D}_2 = \left(\tilde{D}^{X_b x J(a,b)}\right)_{a,b \in \text{ob } \mathcal{J}}$$

Note that $\check{D}_1$ is a homotopy between $s\check{d}_1$ and 1, and $\check{D}_2$ is a homotopy between $s\check{d}_2$ and 1. The commutative diagram in 3.10.8 gives us the following corollary.

Corollary. $\tilde{D}$ induces a homotopy $\bar{D}:\bar{\Delta}(1)\otimes C^*_{\mathcal{J}}(X;\ \kappa) \longrightarrow C^*_{\mathcal{J}}(X;\ \kappa)$ between $s\check{d}$ and 1.

Our goal is to define a submodule generated by the covering.

3.10.10 Definition. We will us the following notation: $u^a_A = \underline{\text{Hom}}(u^A_a, 1_{K_a})$, $u^{ab}_A = \underline{\text{Hom}}(u^A_{ab}, 1_{K_a})$, $i^a_{\mathcal{U}} = \underline{\text{Hom}}(i^{\mathcal{U}}_a, 1_{K_a})$, $i^{ab}_{\mathcal{W}} = \underline{\text{Hom}}(i^{\mathcal{W}}_{ab}, 1_{K_a})$, and $\phi^{\mathcal{U}}_{ab} = \underline{\text{Hom}}(S\mathcal{U}_{ab}, 1_{K_a})$.

Taking $\underline{\text{Hom}}$ of chain complexes in 3.9.10 with κ_a gives us the next lemma.

3.10.11 Lemma.

(1) $u^a_A \circ i^a_{\mathcal{U}} = i^{A\#}_a$, $u^{ab}_A \circ i^{ab}_{\mathcal{W}} = (i^A_b \underline{x} 1_{J(a,b)})^\#$.

(2) The following diagram commutes:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(S_\#(A_a),\mathbb{K}_a) \arrow[r, "\phi^A_{ab}"] & \underline{\mathrm{Hom}}(S_\#(A_b\underline{x}\mathcal{J}(a,b)),\mathbb{K}_a) \\
\underline{\mathrm{Hom}}(S_\#(\mathcal{U}_a),\mathbb{K}_a) \arrow[r, "\phi^{\mathcal{U}}_{ab}"'] \arrow[u, "u^a_A"] & \underline{\mathrm{Hom}}(S_\#(X_b\underline{x}\mathcal{J}(a,b)),\mathbb{K}_a) \arrow[u, "u^{ab}_A"'] \\
\underline{\mathrm{Hom}}(S_\#(X_a),\mathbb{K}_a) \arrow[r, "\phi^X_{ab}"'] \arrow[u, "i^a_{\mathcal{U}}"] \arrow[uu, bend left=60, "i^{A\#}_a"] & \underline{\mathrm{Hom}}(S_\#(X_b\underline{x}\mathcal{J}(a,b)),\mathbb{K}_a) \arrow[u, "i^{ab}_{\mathcal{W}}"'] \arrow[uu, bend right=60, "(i^A_b\underline{x}1)^\#"']
\end{tikzcd}$$

Let a,b $\in$ ob $\mathcal{J}$. Define $\psi^{\mathcal{U}}_{ab} = \underline{\mathrm{Hom}}(S\mathcal{U}_{ab}, 1_{\mathbb{K}_a})$.

3.10.12 Lemma. The following diagram commutes.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(S_\#(A_b),\mathbb{K}_b) \arrow[r, "\psi^A_{ab}"] & \underline{\mathrm{Hom}}(S_\#(A_b\underline{x}\mathcal{J}(a,b)),\mathbb{K}_a) \\
\underline{\mathrm{Hom}}(S_\#(\mathcal{U}_b),\mathbb{K}_b) \arrow[r, dashed, "\psi^{\mathcal{U}}_{ab}"'] \arrow[u, "u^b_A"] & \underline{\mathrm{Hom}}(S_\#(X_b\underline{x}\mathcal{J}(a,b)),\mathbb{K}_a) \arrow[u, "u^{ab}_A"'] \\
\underline{\mathrm{Hom}}(S_\#(X_a),\mathbb{K}_a) \arrow[r, "\psi^X_{ab}"'] \arrow[u, "i^b_{\mathcal{U}}"] \arrow[uu, bend left=60, "i^{A\#}_a"] & \underline{\mathrm{Hom}}(S_\#(X_b\underline{x}\mathcal{J}(a,b)),\mathbb{K}_a) \arrow[u, "i^{ab}_{\mathcal{W}}"'] \arrow[uu, bend right=60, "(i^A_b\underline{x}1)^\#"']
\end{tikzcd}$$

3.10.13 Definition. Let p_c, $c \in$ ob $\mathcal{J}$, be the canonical projections from $\prod_a \underline{\mathrm{Hom}}(S_\#(\mathcal{U}_a),\mathbb{K}_a)$ to $\underline{\mathrm{Hom}}(S_\#(\mathcal{U}_c),\mathbb{K}_c)$. Define cochain maps $\phi^{\mathcal{U}} = \prod_{a,b} \phi^{\mathcal{U}}_{ab} p_a$

and $\psi^{\mathcal{U}} = \prod_{a,b} \psi^{U}_{a\,b} p_b$. Define $\Psi^{\mathcal{U}} = \phi^{\mathcal{U}} - \psi^{\mathcal{U}}$, $\check{u}_{A1} = (u_A^{a})_{a \in \mathrm{ob}\,\mathcal{J}}$, $\check{u}_{A2} = (u_A^{ab})_{a,b \in \mathrm{ob}\,\mathcal{J}}$, $\check{i}_{\mathcal{U}1} = (i^{a}_{\mathcal{U}})_{a \in \mathrm{ob}\,\mathcal{J}}$ and $\check{i}_{\mathcal{U}2} = (i^{a\,b}_{\mathcal{U}})_{a,b \in \mathrm{ob}\,\mathcal{J}}$.

3.10.14 Lemma. (1) The following diagram commutes:

$$\begin{tikzcd}
\prod_a \underline{\mathrm{Hom}}(\mathcal{S}_\#(A_a), \mathbb{K}_a) \arrow[r, "\Psi^A"] & \prod_{b,c} \underline{\mathrm{Hom}}(\mathcal{S}_\#(A_c \underline{x} \mathcal{J}(b,c)), \mathbb{K}_b) \\
\prod_a \underline{\mathrm{Hom}}(\mathcal{S}_\#(\mathcal{U}_a), \mathbb{K}_a) \arrow[r, "\Psi^{\mathcal{U}}"'] \arrow[u, "\check{u}_{A\,1}"] & \prod_{b,c} \underline{\mathrm{Hom}}(\mathcal{S}_\#(X_c \underline{x} \mathcal{J}(b,c)), \mathbb{K}_b) \arrow[u, "\check{u}_{A2}"'] \\
\prod_a \underline{\mathrm{Hom}}(\mathcal{S}_\#(X_a), \mathbb{K}_a) \arrow[r, "\Psi^X"'] \arrow[u, "\check{i}_{\mathcal{U}1}"] \arrow[uu, bend left=60, "(\check{i}_{(X,A)})_1"] & \prod_{b,c} \underline{\mathrm{Hom}}(\mathcal{S}_\#(X_c \underline{x} \mathcal{J}(b,c)), \mathbb{K}_b) \arrow[u, "\check{i}_{\mathcal{U}2}"'] \arrow[uu, bend right=60, "(\check{i}_{(X,A)})_2"']
\end{tikzcd}$$

(2) $\check{u}_{A1} \circ \check{i}_{\mathcal{U}1} = (\check{i}_{(X,A)})_1$, $\check{u}_{A2} \circ \check{i}_{\mathcal{U}2} = (\check{i}_{(X,A)})_2$.

Proof. The claim follows by combining 3.10.11 and 3.10.12.

3.10.15 Definition.

(1) Define $C^*_{\mathcal{J}}(\mathcal{U}; \mathbb{K}) = \ker(\Psi^{\mathcal{U}})$.

(2) Denote by $i_{\mathcal{U}}$ a chain map induced by the pair $(\check{i}_{\mathcal{U}1}, \check{i}_{\mathcal{U}2})$.

(3) Denote by $i^{A\#}$ a chain map induced by the pair $((\check{i}_{(X,A)})_1, (\check{i}_{(X,A)})_2)$.

(4) Denote by u_A a chain map induced by the pair $(\check{u}_{A1}, \check{u}_{A2})$.

3.10.16 Corollary.

(1) $C^*_{\mathcal{J}}(X; \mathbb{K}) \xrightarrow{i_{\mathcal{U}}} C^*_{\mathcal{J}}(\mathcal{U}; \mathbb{K}) \xrightarrow{u_A} C^*_{\mathcal{J}}(A; \mathbb{K})$.

(2) $u_A \circ i_{\mathcal{U}} = (i_{(X,A)})^*$.

3.10.17 Now, we define a retraction and a subdivision operator determined by the covering.

(1) Let $a \in$ ob J. Define the subdivision homotopy $R^a:\underline{\mathrm{Hom}}(S_\#(X_a), K_a) \longrightarrow \underline{\mathrm{Hom}}(S_\#(X_a), K_a)$ by $R^a(f)_{p,q}(\sigma) = f_{p,q}(R_a(\sigma))$. We refer the reader to 3.9.17 for definitions of R_a, τ_a, RJ_{ab}, and τ_{ab}. Define $\tau^a:\underline{\mathrm{Hom}}(S(\mathcal{U}_a), K_a) \longrightarrow \underline{\mathrm{Hom}}(S_\#(X_a), K_a)$ by $\tau^a(f)_{p,q}(\sigma) = f_{p,q}(\tau_a(\sigma))$.

Recall that $i^a_{\mathcal{U}}:\underline{\mathrm{Hom}}(S_\#(X_a), K_a) \longrightarrow \underline{\mathrm{Hom}}(S(\mathcal{U}_a), K_a)$ defined by $i^a_{\mathcal{U}}(f)_{p,q}(\sigma) = f_{p,q}(i^{\mathcal{U}}_a(\sigma))$ is induced by the inclusion $i^{\mathcal{U}}_a:S(\mathcal{U}_a) \longrightarrow S_\#(X_a)$. Then $i^a_{\mathcal{U}} \circ \tau^a = 1$ and $R^a:\tau^a \circ i^a_{\mathcal{U}} \simeq 1$ are dual formulas to such in 3.9.18(1).

There is a cochain homotopy map $\tilde{R}^a:\bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(X_a), K_a) \longrightarrow \underline{\mathrm{Hom}}(S_\#(X_a), K_a)$ defined by $\tilde{R}^a(\{0\}\otimes f) = (\tau^a \circ i^a_{\mathcal{U}})(f)$, $\tilde{R}^a(\{1\}\otimes f) = f$, and $\tilde{R}^a(\Omega\otimes f) = R^a(f)$. Note that $\tilde{R}^a(\omega\otimes f)_{p,q}(\sigma) = f_{p,q}(\tilde{R}_a(\omega\otimes\sigma))$.

(2) Let $a,b \in$ ob J. Define $RJ^{ab}:\underline{\mathrm{Hom}}(S_\#(X_b \times \underline{J}(a,b)), K_a) \longrightarrow \underline{\mathrm{Hom}}(S_\#(X_b \times \underline{J}(a,b)), K_a)$ by $RJ^{ab}(f)_{p,q}(\sigma,\eta) = f_{p,q}(RJ_{ab}(\sigma,\eta))$. Define $\tau^{ab}:\underline{\mathrm{Hom}}(S(\mathcal{UJ}_{ab}), K_a) \longrightarrow \underline{\mathrm{Hom}}(S_\#(X_b \times \underline{J}(a,b)), K_a)$ by $\tau^{ab}(f)_{p,q}(\sigma,\eta) = f_{p,q}(\tau_{ab}(\sigma,\eta))$.

Recall that $i^{ab}_{\mathcal{UJ}}:\underline{\mathrm{Hom}}(S_\#(X_b \times \underline{J}(a,b)), K_a) \longrightarrow \underline{\mathrm{Hom}}(S(\mathcal{UJ}_{ab}), K_a)$ defined by $i^{ab}_{\mathcal{UJ}}(f)_{p,q}(\sigma,\eta) = f_{p,q}(i^{\mathcal{UJ}}_{ab}(\sigma,\eta))$ is induced by the inclusion $i^{\mathcal{UJ}}_{ab}:S(\mathcal{UJ}_{ab}) \longrightarrow S_\#(X_b \times \underline{J}(a,b))$. Then $i^{ab}_{\mathcal{UJ}} \circ \tau^{ab} = 1$ and $RJ^{ab}:\tau^{ab} \circ i^{ab}_{\mathcal{UJ}} \simeq 1$ are dual formulas to those in 3.9.18(2).

There is a cochain homotopy map $\tilde{R}J^{ab}:\bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_\#(X_b \times \underline{J}(a,b)), K_a) \longrightarrow \underline{\mathrm{Hom}}(S_\#(X_b \times \underline{J}(a,b)), K_a)$ defined by $\tilde{R}J^{ab}(\{0\}\otimes f) = (\tau^{ab} \circ i^{ab}_{\mathcal{UJ}})(f)$, $\tilde{R}J^{ab}(\{1\}\otimes f) = f$, and $\tilde{R}J^{ab}(\Omega\otimes f) = RJ^{ab}(f)$. Note that $\tilde{R}J^{ab}(\omega\otimes f)_{p,q}(\sigma) =$

$f_{p,q}(\tilde{R}_{ab}(\omega\otimes\sigma))$.

3.10.18 Lemma (for ϕ's). Let $a,b \in \mathrm{ob}\,J$. The following diagram commutes:

$$\begin{array}{ccc}
\underline{\mathrm{Hom}}(S_{\#}(X_a),\, K_a) & \xrightarrow{\phi^X_{ab}} & \underline{\mathrm{Hom}}(S_{\#}(X_b \underline{x} J(a,b)),\, K_a) \\
\uparrow{\scriptstyle \tilde{R}^a} & & \uparrow{\scriptstyle \tilde{R}J^{ab}} \\
\bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_{\#}(X_a),\, K_a) & \xrightarrow[1\otimes\phi^X_{ab}]{} & \bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_{\#}(X_b \underline{x} J(a,b)),\, K_a)\,.
\end{array}$$

Proof. We shall show that $\phi^X_{ab}(\tilde{R}^a(\omega\otimes f)) = \tilde{R}J^{ab}((1\otimes\phi^X_{ab})(\omega\otimes f))$.

We have $\phi^X_{ab}(\tilde{R}^a(\omega\otimes f))_{p,q}(\sigma,\eta) = \tilde{R}^a(\omega\otimes f)_{p,q}(\tilde{X}_{ab\#}(\sigma,\eta))$. Since $\tilde{R}^a(\omega\otimes f)_{p,q}(\sigma) = f_{p,q}(\tilde{R}_a(\omega\otimes\sigma))$, $\phi^X_{ab}(\tilde{R}^a(\omega\otimes f))_{p,q}(\sigma,\eta) = f_{p,q}(\tilde{R}_a(\omega\otimes\tilde{X}_{ab\#}(\sigma,\eta)))$.

On the other side the identity $\tilde{R}J^{ab}(\omega\otimes f)_{p,q}(\sigma,\eta) = f_{p,q}(\tilde{R}_{ab}(\omega\otimes(\sigma,\eta)))$ gives us $\tilde{R}J^{ab}((1\otimes\phi^X_{ab})(\omega\otimes f))_{p,q}(\sigma,\eta) = \tilde{R}J^{ab}(\omega\otimes\phi^X_{ab}(f))_{p,q}(\sigma,\eta) = \phi^X_{ab}(f)_{p,q}(\tilde{R}_{ab}(\omega\otimes(\sigma,\eta))) = f_{p,q}(\tilde{X}_{ab\#}(\tilde{R}_{ab}(\omega\otimes(\sigma,\eta))))$. Since in the course of proving of 3.9.18 we obtained $\tilde{X}_{ab\#}\left(\tilde{R}J_{ab}(\omega\otimes(\sigma,\eta))\right) = \tilde{R}_a(\omega\otimes\tilde{X}_{ab\#}(\sigma,\eta))$, the claim follows.

3.10.19 Lemma (for ψ's). Let $a,b \in \mathrm{ob}\,J$. The following diagram commutes:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(S_{\#}(X_b),\ \mathbb{K}_b) & \xrightarrow{\psi^X_{ab}} & \underline{\mathrm{Hom}}(S_{\#}(X_b \underline{x} J(a,b)),\ \mathbb{K}_a) \\ \uparrow{\scriptstyle \tilde{R}^b} & & \uparrow{\scriptstyle \tilde{R}J^{ab}} \\ \bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_{\#}(X_b),\ \mathbb{K}_b) & \xrightarrow[1\otimes\psi^X_{ab}]{} & \bar{\Delta}^{op}(1)\otimes\underline{\mathrm{Hom}}(S_{\#}(X_b \underline{x} J(a,b)),\ \mathbb{K}_a) \end{array}.$$

Proof. We shall show that $\psi^X_{ab}(\tilde{R}^a(\omega\otimes f)) = \tilde{R}J^{ab}((1\otimes\psi^X_{ab})(\omega\otimes f))$.

We have $\psi^X_{ab}(\tilde{R}^a(\omega\otimes f))_{p,q}(\sigma,\eta) = \check{\mathbb{K}}(\eta)(\tilde{R}^a(\omega\otimes f)_{p,q}(\sigma)) = \check{\mathbb{K}}(\eta)(f_{p,q}(\tilde{R}_a(\omega\otimes\sigma)))$. On the other hand $\tilde{R}J^{ab}((1\otimes\psi^X_{ab})(\omega\otimes f))_{p,q}(\sigma,\eta) = \tilde{R}J^{ab}(\omega\otimes\psi^X_{ab}(f))_{p,q}(\sigma,\eta) = \psi^X_{ab}(f)_{p,q}(\tilde{R}_{ab}(\omega\otimes(\sigma,\eta)))$. Let $\omega = \{0\}$. Then $\psi^X_{ab}(\tilde{R}^a(\{0\}\otimes f))_{p,q}(\sigma,\eta) = \check{\mathbb{K}}(\eta)(f_{p,q}(\tilde{R}_a(\{0\}\otimes\sigma))) = \check{\mathbb{K}}(\eta)(f_{p,q}((i^{\mathcal{U}}_a \circ \tau_a)(\sigma))) =$

$$\check{\mathbb{K}}(\eta)(f_{p,q}(\tau_a(\sigma))) = \check{\mathbb{K}}(\eta)\Big[f_{p,q}\Big(\sum_{i=0}^{|\sigma|}\ \sum_{j=m(\sigma^{(i)})}^{m(\sigma)-1} D(sd^j(\sigma^{(i)})) + sd^{m(\sigma)}(\sigma)\Big)\Big] =$$

$$\sum_{i=0}^{|\sigma|}\ \sum_{j=m(\sigma^{(i)})}^{m(\sigma)-1} \check{\mathbb{K}}(\eta)\Big[f_{p,q}(D(sd^j(\sigma^{(i)})))\Big] + \check{\mathbb{K}}(\eta)\Big[f_{p,q}(sd^{m(\sigma)}(\sigma))\Big].$$

On the other hand we have

$$\tilde{R}J^{ab}((1\otimes\psi^X_{ab})(\{0\}\otimes f))_{p,q}(\sigma,\eta) = \psi^X_{ab}(f)_{p,q}(\tilde{R}_{ab}(\{0\}\otimes(\sigma,\eta))) =$$

$$\psi^X_{ab}(f)_{p,q}((i^{\mathcal{W}}_{ab}\circ\tau_{ab})(\sigma,\eta)) = \psi^X_{ab}(f)_{p,q}(\tau_{ab}(\sigma,\eta)) =$$

$$\psi^X_{ab}(f)_{p,q}\Big(\sum_{i=0}^{|(\sigma,\eta)|}\ \sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} D(sd^j(\sigma^{(i)},\eta^{(i)})) + sd^{m(\sigma,\eta)}(\sigma,\eta)\Big] =$$

$$\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \psi^X_{ab}(f)_{p,q}\left[D_{XbxJ(a,b)}\left(sd_{XbxJ(a,b)}{}^{j}(\sigma^{(i)},\eta^{(i)})\right)\right] +$$

$$\psi^X_{ab}(f)_{p,q}\left(sd_{XbxJ(a,b)}{}^{m(\sigma,\eta)}(\sigma,\eta)\right) =$$

$$\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \left[\tilde{D}^{XbxJ(a,b)}(\Omega\otimes\psi^X_{ab}(f))\right]_{p,q}\left(sd^{j}(\sigma^{(i)},\eta^{(i)})\right) +$$

$$\left[\left(sd^{XbxJ(a,b)}\right)^{m(\sigma,\eta)}\left(\psi^X_{ab}(f)\right)\right]_{p,q}(\sigma,\eta) =$$

$$\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \left[\tilde{D}^{XbxJ(a,b)}(1\otimes\psi^X_{ab})(\Omega\otimes f)\right]_{p,q}\left(sd^{j}(\sigma^{(i)},\eta^{(i)})\right) +$$

$$\left[\left(sd^{XbxJ(a,b)}\right)^{m(\sigma,\eta)}\left(\psi^X_{ab}(f)\right)\right]_{p,q}(\sigma,\eta) =$$

$$\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \left[\left(sd^{XbxJ(a,b)}\right)^{j}\left(\tilde{D}^{XbxJ(a,b)}(1\otimes\psi^X_{ab})(\Omega\otimes f)\right)\right]_{p,q}(\sigma^{(i)},\eta^{(i)}) +$$

$$\left[\left(sd^{XbxJ(a,b)}\right)^{m(\sigma,\eta)}\left(\psi^X_{ab}(f)\right)\right]_{p,q}(\sigma,\eta). \quad .$$

Using 3.10.7 and next 3.10.3 we continue

$$\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \left[\left(sd^{XbxJ(a,b)}\right)^{j}\left(\psi^X_{ab}\circ\tilde{D}^{Xb}\right)(\Omega\otimes f)\right]_{p,q}(\sigma^{(i)},\eta^{(i)}) +$$

$$\left[\left(sd^{XbxJ(a,b)}\right)^{m(\sigma,\eta)}\left(\psi^X_{ab}(f)\right)\right]_{p,q}(\sigma,\eta) =$$

$$\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \left[\psi^X_{ab}\left(\left(sd^{Xb}\right)^{j}\circ\tilde{D}^{Xb}\right)(\Omega\otimes f)\right]_{p,q}(\sigma^{(i)},\eta^{(i)}) +$$

$$\left[\psi^X_{ab}\left(\left(sd^{Xb}\right)^{m(\sigma,\eta)}(f)\right)\right]_{p,q}(\sigma,\eta) =$$

$$\sum_{i=0}^{|(\sigma,\eta)|}\sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \check{\kappa}(\eta^{(i)})\left[\left(\left(\left(sd^{Xb}\right)^{j}\circ\tilde{D}^{Xb}\right)(\Omega\otimes f)\right)_{p,q}(\sigma^{(i)})\right] +$$

$$\check{\kappa}(\eta)\left[\left((sd^{Xb})^{m(\sigma,\eta)}(f)\right)_{p,q}(\sigma)\right] =$$

$$\sum_{i=0}^{|(\sigma,\eta)|}\ \sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \check{\kappa}(\eta^{(i)})\left[\left(\tilde{D}^{Xb}(\Omega\otimes f)\right)_{p,q}\left(sd^{j}(\sigma^{(i)})\right)\right] +$$

$$\check{\kappa}(\eta)\left[f_{p,q}\left(sd^{m(\sigma,\eta)}(\sigma)\right)\right] =$$

$$\sum_{i=0}^{|(\sigma,\eta)|}\ \sum_{j=m(\sigma^{(i)},\eta^{(i)})}^{m(\sigma,\eta)-1} \check{\kappa}(\eta^{(i)})\left[f_{p,q}\left(D(sd^{j}(\sigma^{(i)}))\right)\right] + \check{\kappa}(\eta)\left[f_{p,q}\left(sd^{m(\sigma,\eta)}(\sigma)\right)\right] =$$

Since $\check{\kappa}(\eta^{(i)}) = \check{\kappa}(\eta)$ and $m(\sigma,\eta) = m(\sigma)$, the required equality holds.

3.10.20 Proposition. The following diagram commutes:

$$\begin{array}{ccc}
\prod_{a} \underline{\mathrm{Hom}}(\mathcal{S}_{\#}(X_a), \kappa_a) & \xrightarrow{\Psi^X} & \prod_{b,c} \underline{\mathrm{Hom}}(\mathcal{S}_{\#}(X_c \underline{x} \mathcal{J}(b,c)), \kappa_b) \\
\uparrow (\tilde{R}^a) & & \uparrow (\tilde{R}\mathcal{J}^{ab}) \\
\bar{\Delta}^{op}(1)\otimes\left(\prod_{a} \underline{\mathrm{Hom}}(\mathcal{S}_{\#}(X_a), \kappa_a)\right) & \xrightarrow[1\otimes\Psi^X]{} & \bar{\Delta}^{op}(1)\otimes\left(\prod_{b,c} \underline{\mathrm{Hom}}(\mathcal{S}_{\#}(X_c \underline{x} \mathcal{J}(b,c)), \kappa_b)\right)
\end{array}.$$

Proof. The claim follows by combining 3.10.18 and 3.10.19.

3.10.21 Definition. Denote $\check{R}_1 = (\tilde{R}^a)_{a\in \mathrm{ob}\ \mathcal{J}}$, $\check{R}_1 = (\tilde{R}\mathcal{J}^{ab})_{a,b\in \mathrm{ob}\ \mathcal{J}}$. Define

$$\check{\tau}_1 = (\tau^a)_{a\in \mathrm{ob}\ \mathcal{J}}: \prod_{a} \underline{\mathrm{Hom}}(\mathcal{S}_{\#}(\mathcal{U}_a), \kappa_a) \longrightarrow \prod_{a} \underline{\mathrm{Hom}}(\mathcal{S}_{\#}(X_a), \kappa_a), \text{ and}$$

$$\check{\tau}_2 = (\tau^{ab})_{a,b\in \mathrm{ob}\ \mathcal{J}}: \prod_{a,b} \underline{\mathrm{Hom}}(\mathcal{S}(\mathcal{U}\mathcal{J}_{ab}), \kappa_a) \longrightarrow \prod_{a,b} \underline{\mathrm{Hom}}(\mathcal{S}_{\#}(X_b \underline{x} \mathcal{J}(a,b)).$$

3.10.22 Proposition.

(1) $\check{R}_1$ is a homotopy between $\check{\tau}_1 \circ \overset{\curlyvee}{i}_{\mathcal{U}1}$ and 1, moreover $\overset{\curlyvee}{i}_{\mathcal{U}1} \circ \check{\tau}_1 = 1$.

(2) $\check{R}_2$ is a homotopy between $\check{\tau}_2 \circ \overset{\curlyvee}{i}_{\mathcal{U}2}$ and 1, moreover $\hat{i}_{\mathcal{U}2} \circ \check{\tau}_2 = 1$.

(3) τ induces the following commutative square of cochain maps.

$$
\begin{array}{ccc}
\prod_{a} \underline{\mathrm{Hom}}(S_{\#}(X_a), \kappa_a) & \xrightarrow{\Psi^{X}} & \prod_{b,c} \underline{\mathrm{Hom}}(S_{\#}(X_c \underline{x} J(b,c)), \kappa_b) \\
\Big\uparrow \check{\tau}_1 & & \Big\uparrow \check{\tau}_2 \\
\prod_{a} \underline{\mathrm{Hom}}(S(\mathcal{U}_a), \kappa_a) & \xrightarrow[\Psi^{\mathcal{U}}]{} & \prod_{b,c} \underline{\mathrm{Hom}}(S(\mathcal{U}J_{ab}), \kappa_b)
\end{array}
$$

Proof. The claims follow directly from the definitions of $\check{R}_1$ and $\check{R}_2$ by checking directly and by the commutativity of the diagram in 3.10.20.

3.10.23 Remark. We summarize properties of $\check{R}_1$ and $\check{R}_2$ in the following commutative diagram.

$$
\begin{tikzcd}
\prod_{a} \underline{\mathrm{Hom}}(S_{\#}(X_{\bullet}), \mathbb{K}_a) \arrow[r, "\Psi^X"] & \prod_{b,c} \underline{\mathrm{Hom}}(S_{\#}(X_c \underline{x} J(b,c)), \mathbb{K}_b) \\
\bar{\Delta}^{op}(1)\otimes\left(\prod_{a} \underline{\mathrm{Hom}}(S_{\#}(X_{\bullet}), \mathbb{K}_a)\right) \arrow[r, "1\otimes\Psi^X"'] \arrow[u, "(\tilde{R}^{\bullet})"] & \bar{\Delta}^{op}(1)\otimes\left(\prod_{b,c} \underline{\mathrm{Hom}}(S_{\#}(X_c \underline{x} J(b,c)), \mathbb{K}_b)\right) \arrow[u, "(\tilde{R}J^{ab})"'] \\
\prod_{a} \underline{\mathrm{Hom}}(S_{\#}(X_{\bullet}), \mathbb{K}_a) \arrow[r, "\Psi^X"'] \arrow[u] & \prod_{b,c} \underline{\mathrm{Hom}}(S_{\#}(X_c \underline{x} J(b,c)), \mathbb{K}_b) \arrow[u]
\end{tikzcd}
$$

From the commutativity of the diagram in 3.10.20 we have the following proposition.

3.10.24 Proposition. $\tilde{R}$ induces a homotopy $R:\bar{\Delta}^{op}(1)\otimes C^*_{\mathcal{J}}(X;\ \mathbb{K}) \longrightarrow C^*_{\mathcal{J}}(X;\ \mathbb{K})$ between $\check{\tau}\circ i_{\mathcal{U}}$ and 1, and $i_{\mathcal{U}}\circ\check{\tau} = 1$; therefore $i_{\mathcal{U}}:C^*_{\mathcal{J}}(X;\ \mathbb{K}) \longrightarrow C^*_{\mathcal{J}}(\mathcal{U};\ \mathbb{K})$ is a cochain equivalence.

3.10.25 Theorem (EXCISION). $i_{\mathcal{U}}^*:\mathbb{H}^*_{\mathcal{J}}(X;\ \mathbb{K}) \longrightarrow \mathbb{H}^*_{\mathcal{J}}(\mathcal{U};\ \mathbb{K})$ is an isomorphism.

As it was with singular $\mathcal{J}$-homology we extend without proof the claim of Theorem 3.10.25 to the case of a saturated pair (X, B) with a covering restricting well to B (see 3.9.27). We denote by $i_{\mathcal{U}(X,B)}:C^*_{\mathcal{J}}(X,B;\ \mathbb{K}) \longrightarrow C^*_{\mathcal{J}}(\mathcal{U},\mathcal{U}\cap B;\ \mathbb{K})$ the induced inclusion morphism between $C^*_{\mathcal{J}}(X,B;\ \mathbb{K})$ and the complex of $\mathcal{U}$-small singular $\mathcal{J}$-cochains (compare with $i_{\mathcal{U}}$ in 3.10.14(2)).

3.10.26 Theorem (GENERALIZED EXCISION). $i_{\mathcal{U}(X,B)}{}^*: H^*_{\mathcal{J}}(X,B;\ \mathbb{K}) \longrightarrow H^*_{\mathcal{J}}(\mathcal{U},\mathcal{U}\cap B;\ \mathbb{K})$ is an isomorphism.

3.10.27 Let (X, B) be a saturated $\mathcal{J}$-pair and let U be an open saturated subdiagram of X such that $\bar{U} \subseteq B$. Let $\mathcal{V} = \{B,\ X\text{-}U\}$. As was shown in 3.9.28 $\mathcal{V}$ is a covering of X and it restricts well to B.

Similar to 3.9.29 we have the following.

3.10.28 Lemma.

(1) There exists an natural isomorphism $C^*_{\mathcal{J}}(X\text{-}U,B\text{-}U;\ \mathbb{K}) \longrightarrow C^*_{\mathcal{J}}(\mathcal{U},\mathcal{U}\cap B;\ \mathbb{K})$.

(2) $i_{\mathcal{V}(X,B)}: C^*_{\mathcal{J}}(X,B;\ \mathbb{K}) \longrightarrow C^*_{\mathcal{J}}(\mathcal{U},\mathcal{U}\cap B;\ \mathbb{K})$ (see 3.10.25) under the isomorphism in (1) can be identified as a morphism induced by the inclusion $u:(X\text{-}U,B\text{-}U) \subseteq (X,\ B)$.

Similar to 3.9.29 we have the following Theorem.

3.10.29 Theorem (EXCISION). Let (X,A) be a saturated pair. Let U be an open saturated subdiagram of X whose closure $\bar{U}$ is contained in the interior of A. Then, the inclusion morphism $u:(X\text{-}U,A\text{-}U) \longrightarrow (X,A)$ is admissible and it induces an isomorphism $u^*: H^*_{\mathcal{J}}(X,A) \cong H^*_{\mathcal{J}}(X\text{-}U,A\text{-}U)$.

3.11 Dimension Axiom for Singular J-homology

3.11.1 Let $j \in \text{ob } J$. Recall that j induces a diagram $D_j = J(\_,j)$. We shall determine $H_*^J(J(\_,j); \mathbb{M})$. As the first step we shall determine the J-singular chain complex $C_*^J(J(\_,j); \mathbb{M})$. Denote by $\hat{\mathbb{Z}}$ the chain complex defined by $\hat{\mathbb{Z}}_n = \mathbb{Z}$ for $n \geq 0$ and $\hat{\mathbb{Z}}_n = 0$ for $n < 0$, with the differential $\partial_n = 0$ for n odd and $\partial_n = 1$ for n even, that is $\hat{\mathbb{Z}}$ is the same as the singular complex of a one point space. Denote by $\mathbf{1}_n$, $n \geq 0$, the generator of $\hat{\mathbb{Z}}_n$. Then

$$\partial \mathbf{1}_n = \sum_{i=0}^{n} (-1)^i \mathbf{1}_{n-1} = \begin{cases} \mathbf{1}_{n-1} & \text{if n even} \\ 0 & \text{if n odd} \end{cases}.$$

We now prove the following proposition.

3.11.2 Proposition. The J-singular chain complex $C_*^J(J(\_,j); \mathbb{M})$ is isomorphic to $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{M}_j$.

Proof. The proof is given in 3.11.9.

3.11.3 Remark The J-singular chain complex $C_*^J(J(\_,j); \mathbb{M})$ was defined via the cokernel of $\Phi^j = \alpha^j - \beta^j$, where superscript j denotes the diagram $D_j = J(\_,j)$, and therefore it is a coequalizer object of maps α^j and β^j.

$$\sum_{a,b} S_\#(J(a,j)\underline{x}J(a,b)) \otimes_{\mathbb{Z}} \mathbb{M}_a \underset{\beta^j}{\overset{\alpha^j}{\rightrightarrows}} \sum_{c} S_\#(J(c,j)) \otimes_{\mathbb{Z}} \mathbb{M}_c \xrightarrow{\mu} C_*^J(J(\_,j); \mathbb{M}$$

Recall that $\alpha^j | S_\#(J(b,j)\underline{x}J(a,b)) \otimes \mathbb{M}_a = \alpha^j_{ab}$ and $\beta^j | S_\#(J(b,j)\underline{x}J(a,b)) \otimes \mathbb{M}_a = \beta^j_{ab}$.

3.11.4 Proposition. Let $\{\mu_a : S_\#(J(a,j))\otimes\mathbf{M}_a \longrightarrow C_*^J(J(\_,j);\ \mathbf{M}) \mid a \in \text{ob } J\}$ be a family of components of the coequalizer map μ. Let a,b $\in$ ob J. Then each diagram

$$\begin{tikzcd}
S_\#(J(b,j) \underline{\times} J(a,b))\otimes\mathbf{M}_a \arrow[r, "\alpha^j_{ab}"] \arrow[d, "\beta^j_{ab}"'] & S_\#(J(a,j))\otimes\mathbf{M}_a \arrow[d, "\mu_a"] \\
S_\#(J(b,j))\otimes\mathbf{M}_b \arrow[r, "\mu_b"'] & C_*^J(J(_,j);\ \mathbf{M})
\end{tikzcd}$$

commutes. Moreover, the family $\{\mu_a\}_{a \in \text{ob } J}$ is universal in the sense that for every family $\{\lambda_a : S_\#(J(a,j))\otimes\mathbf{M}_a \longrightarrow \mathbf{A} \mid a \in \text{ob } J\}$ of chain maps which ensure the commutativity of the above diagram, there is a unique chain map

$$h : C_*^J(J(\_,j);\ \mathbf{M}) \longrightarrow \mathbf{A}$$

such that $\lambda_a = h\circ\mu_a$, a $\in$ ob J.

$$\begin{tikzcd}
S_\#(J(b,j) \underline{\times} J(a,b))\otimes\mathbf{M}_a \arrow[r, "\alpha^j_{ab}"] \arrow[d, "\beta^j_{ab}"'] & S_\#(J(a,j))\otimes\mathbf{M}_a \arrow[d, "\mu_a"] \arrow[dr, "\lambda_a"] & \\
S_\#(J(b,j))\otimes\mathbf{M}_b \arrow[r, "\mu_b"'] \arrow[rr, bend right, "\lambda_b"'] & C_*^J(J(_,j);\ \mathbf{M}) \arrow[r, dotted, "h"] & \mathbf{A}
\end{tikzcd}$$

Proof. The claim follows from the the general properties of a coequalizer diagram in 3.11.3.

3.11.5 Definition. Define a chain map $\lambda_a : S_\#(J(a,j)) \otimes_{\mathbb{Z}} \mathbb{M}_a \longrightarrow \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{M}_j$, $a \in$ ob J, by $\lambda_a(\sigma\otimes m) = 1_n \otimes \check{\mathbb{M}}(\sigma)(m)$. We check that each λ_a is a chain map.

Let $\sigma\otimes m$ be a generator of $_\#(J(a,j))\otimes_{\mathbb{Z}}\mathbb{M}_a$ with $n = |\sigma|$. Then $\lambda\partial(\sigma\otimes m) =$

$$\lambda\Big(\sum_{i=0}^{n}(-1)^i\sigma^{(i)}\otimes m \;+\; (-1)^n\sigma\otimes\partial m\Big) \;=\; \sum_{i=0}^{n}(-1)^i 1_{n-1}\otimes\check{\mathbb{M}}(\sigma^{(i)})(m) \;+\; (-1)^n 1_n\otimes\check{\mathbb{M}}(\sigma)(\partial m),$$

and $\partial\lambda(\sigma\otimes m) = \partial\big(1_n\otimes\check{\mathbb{M}}(\sigma)(m)\big) = \partial 1_n\otimes\check{\mathbb{M}}(\sigma)(m) + (-1)^n 1_n\otimes\partial\check{\mathbb{M}}(\sigma)(m) =$

$$\sum_{=0}^{n}(-1)^i 1_{n-1}\otimes\check{\mathbb{M}}(\sigma)(m) + (-1)^n 1_n\otimes\partial\check{\mathbb{M}}(\sigma)(m).$$

Since image$(\sigma^{(i)}) \subseteq$ image(σ) and $\check{\mathbb{M}}(\sigma)$ is a chain map, $\check{\mathbb{M}}(\sigma^{(i)}) = \check{\mathbb{M}}(\sigma)$, and $\check{\mathbb{M}}(\sigma)(\partial m) = \partial\check{\mathbb{M}}(\sigma)(m)$. Therefore $\lambda\partial(\sigma\otimes m) = \partial\lambda(\sigma\otimes m)$ as required.

3.11.6 Lemma. Let $a,b \in$ ob J. The following diagram commutes:

$$\begin{tikzcd}
S_\#(J(b,j)\underline{\times}J(a,b))\otimes\mathbb{M}_a \arrow[r, "\alpha^j_{ab}"] \arrow[d, "\beta^j_{ab}"'] & S_\#(J(a,j))\otimes\mathbb{M}_a \arrow[d, "\lambda_a"] \\
S_\#(J(b,j))\otimes\mathbb{M}_b \arrow[r, "\lambda_b"'] & \hat{\mathbb{Z}}\otimes\mathbb{M}_j
\end{tikzcd}$$

Proof. Recall the composition $m_J : J(b,j)\underline{\times}J(a,b) \longrightarrow J(a,j)$, $(f,g) \longrightarrow f\circ g$ (of the small top. category J). Let $(\sigma,\eta)\otimes m$ be a generator of $S_\#(J(b,j)\underline{\times}J(a,b)) \otimes \mathbb{M}_a$. Note that $\alpha^j_{ab}((\sigma,\eta)\otimes m) = ((m_J)_\#\otimes 1_{\mathbb{M}_a})((\sigma,\eta)\otimes m) = (m_J)_\#(\sigma,\eta)\otimes m$ and $\beta^j_{ab}((\sigma,\eta)\otimes m) = \sigma\otimes\check{\mathbb{M}}(\eta)(m)$. Therefore $\lambda_a\alpha^j_{ab}((\sigma,\eta)\otimes m) = \lambda_a((m_J)_\#(\sigma,\eta)\otimes m) = \check{\mathbb{M}}((m_J)_\#(\sigma,\eta))(m)$ and $\lambda_b\beta^j_{ab}((\sigma,\eta)\otimes m) = \lambda_b(\sigma\otimes\check{\mathbb{M}}(\eta)(m)) = \check{\mathbb{M}}(\sigma)(\check{\mathbb{M}}(\eta)(m))$. Denote $\sigma\circ\eta = (m_J)_\#(\sigma,\eta)$. Clearly we need to show that $\check{\mathbb{M}}$

is a "covariant functor"; that is,

$$\check{\mathbb{M}}(\sigma \circ \eta) = \check{\mathbb{M}}(\sigma) \circ \check{\mathbb{M}}(\eta).$$

Recall that $\check{\mathbb{M}}(\omega)$ is determined by any $f \in \text{image}(\omega) \subseteq \mathcal{J}(a,b)$ by the formula $\check{\mathbb{M}}(\omega) = \mathbb{M}(f)$. Let (σ,η) be a singular n–simplex of $\mathcal{J}(b,j)\underline{x}\mathcal{J}(a,b)$. Choose $t_0 \in \Delta_n$. Let $f = \sigma(t_0)$, $g = \eta(t_0)$, and denote $f \circ g = m_{\mathcal{J}}(f,g) = (m_{\mathcal{J}})_{\#}(\sigma,\eta)(t_0)$. Clearly $f \circ g \in \text{image}(\sigma,\eta)$. Then $\check{\mathbb{M}}(\sigma \circ \eta) = \mathbb{M}(f \circ g)$, $\check{\mathbb{M}}(\sigma) = \mathbb{M}(f)$, $\check{\mathbb{M}}(\eta) = \mathbb{M}(g)$. The functor $\mathbb{M}$ satisfies $\mathbb{M}(f \circ g) = \mathbb{M}(f) \circ \mathbb{M}(g)$ and therefore $\check{\mathbb{M}}(\sigma \circ \eta) = \mathbb{M}(f \circ g) = \mathbb{M}(f) \circ \mathbb{M}(g) = \check{\mathbb{M}}(\sigma) \circ \check{\mathbb{M}}(\eta)$ as required.

From 3.11.4 and 3.11.5, we have the following corollary.

3.11.7 Corollary. There is a unique chain map $h{:}C_*^{\mathcal{J}}(\mathcal{J}(\_,j);\ \mathbb{M}) \longrightarrow \hat{\mathbb{Z}} \otimes \mathbb{M}_j$, such that $\lambda_a = h \circ \mu_a$, $a \in \text{ob}\ \mathcal{J}$.

3.11.8 Definition. Here we shall define the inverse of h. For the identity element $id_j \in \mathcal{J}(j,j)$ and the nonnegative integer n, define a singular n–simplex $\omega(\mathbf{1}_n){:}\Delta_n \longrightarrow \mathcal{J}(j,j)$, $t \longrightarrow id_j$. Note that $\partial\omega(\mathbf{1}_n) = \sum_{i=0}^{n}(-1)^i\omega(\mathbf{1}_{n-1})$.

Define a chain map $f{:}\hat{\mathbb{Z}} \otimes \mathbb{M}_j \longrightarrow C_*^{\mathcal{J}}(\mathcal{J}(\_,j);\ \mathbb{M})$ by $f(\mathbf{1}_n \otimes m) = \mu_j(\omega(\mathbf{1}_n) \otimes m)$ for a generator $\mathbf{1}_n \otimes m$. f is a chain map since $\partial f(\mathbf{1}_n \otimes m) = \partial\mu_j(\omega(\mathbf{1}_n) \otimes m) =$

$$\mu_j\partial(\omega(\mathbf{1}_n) \otimes m) = \mu_j\left(\sum_{i=0}^{n}(-1)^i\omega(\mathbf{1}_{n-1}) \otimes m + (-1)^n\omega(\mathbf{1}_n) \otimes \partial m\right) \quad \text{and}$$

$$f\partial(\mathbf{1}_n \otimes m) = f\left(\sum_{i=0}^{n}(-1)^i\mathbf{1}_{n-1} \otimes m + (-1)^n\mathbf{1}_n \otimes \partial m\right) = \sum_{i=0}^{n}(-1)^i f(\mathbf{1}_{n-1} \otimes m) + (-1)^n f(\mathbf{1}_n \otimes \partial m)$$

$$= \sum_{i=0}^{n} (-1)^i \mu_j(\omega(1_{n-1})\otimes m) + (-1)^n \mu_j(\omega(1_n)\otimes \partial m) =$$

$$\mu_j\left[\sum_{i=0}^{n} (-1)^i \omega(1_{n-1})\otimes m + (-1)^n \omega(1_n)\otimes \partial m\right].$$

3.11.9 Proof of Proposition 3.11.2. We shall show that the chain map $h: C_*^J(J(\_,j);\ \mathbb{M}) \longrightarrow \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{M}_j$ is an isomorphism with inverse f. For a generator $1_n \otimes m$ of $\hat{\mathbb{Z}} \otimes \mathbb{M}_j$, we have

$(h \circ f)(1_n\otimes m) = h\ (\mu_j(\omega(1_n)\otimes m)) = (h \circ \mu_j)(\omega(1_n)\otimes m) = \lambda_j(\omega(1_n)\otimes m) = 1_n\otimes\check{\mathbb{M}}(\omega(1_n))(m) = 1_n\otimes\mathbb{M}(id_j)(m) = 1_n\otimes m.$

Therefore $h \circ f = 1$. Note that $C_*^J(J(\_,j);\ \mathbb{M})$ is generated by all elements of the form $\mu_a(\sigma\otimes m)$, and we have $(f \circ h)(\mu_a(\sigma\otimes m)) = f((h \circ \mu_a)(\sigma\otimes m)) = f(\lambda_a(\sigma\otimes m)) = f(1_n\otimes\check{\mathbb{M}}(\sigma)(m)) = \mu_j(\omega(1_n)\otimes\check{\mathbb{M}}(\sigma)(m))$. Since $\alpha^j_{ab}((\omega(1_n),\sigma)\otimes m) = (\omega(1_n)\circ\sigma)\otimes m = \sigma\otimes m$ and $\beta^j_{ab}((\omega(1_n),\sigma)\otimes m) = \omega(1_n)\otimes\check{\mathbb{M}}(\sigma)(m)$, also $\mu_a(\sigma\otimes m) = \mu_j(\omega(1_n)\otimes\check{\mathbb{M}}(\sigma)(m))$. Thus $(f \circ h)(\mu_a(\sigma\otimes m)) = \mu_a(\sigma\otimes m)$, and therefore $f \circ h = 1$.

Immediately from Proposition 2, we have the following corollary.

3.11.10 Corollary.

(1) $H_*^J(J(\_,j);\ \mathbb{M}) \cong H_*(\mathbb{M}_j)$.

(2) If the coefficient chain complex $\mathbb{M}_j$ is concentrated in zero dimension,

with a module M as a zero grading, then $H_*^J(J(\_,j);\ \mathbb{M}) \cong M$.

3.11.11 Theorem (DIMENSION). If the coefficient system $\mathbb{M}$ takes values in **Ab**, then the singular J-homology theory satisfies the Dimension Axiom.

3.12 Dimension Axiom for Singular J-Cohomology

We shall prove the following proposition.

3.12.1 Proposition. The J-singular cochain complex $C_J^*(J(\_,j);\ \mathbb{K})$ is <u>isomorphic</u> to $\underline{\mathrm{Hom}}_Z(\hat{\mathbb{Z}}, \mathbb{K}j)$.

Proof. The proof is given in 3.12.10.

3.12.2 Remark. Recall that the J-singular cochain complex $C_J^*(J(\_,j);\ \mathbb{K})$ is defined via the kernel of $\Psi^j = \phi^j - \psi^j$ (the superscript j denotes $D_j = J(\_,j)$), and therefore it is the equalizer object of the maps ϕ^j and ψ^j

$$\prod_a \underline{\mathrm{Hom}}(S_*(J(a,j)),\ \mathbb{K}a) \ \underset{\psi^j}{\overset{\phi^j}{\rightrightarrows}}\ \prod_{b,c} \underline{\mathrm{Hom}}(S_*(J(c,j)\underline{x}J(b,c)),\ \mathbb{K}b).$$

3.12.3 Proposition. Let

$\{\mu_a:\ |C_J^*(J(\_,j);\ \mathbb{K}) \longrightarrow \underline{\mathrm{Hom}}(S_*(J(a,j)),\ \mathbb{K}a)\,|\,a \in \mathrm{ob}\ J\}$

be a family of components of the equalizer map μ. Let $a,b \in \mathrm{ob}\ J$. Then each diagram

$$\begin{tikzcd}
C^*_J(J(_,j);\, \mathbb{K}) \arrow[r, "\mu_a"] \arrow[d, "\mu_b"'] & \underline{\mathrm{Hom}}(S_\#(J(a,j)),\, \mathbb{K}_a) \arrow[d, "\phi^j_{ab}"] \\
\underline{\mathrm{Hom}}(S_\#(J(b,j)),\, \mathbb{K}_b) \arrow[r, "\psi^j_{ab}"'] & \underline{\mathrm{Hom}}(S_\#(J(b,j)\underline{x}J(a,b)),\, \mathbb{K}_a)
\end{tikzcd}$$

commutes. Moreover, the family $\{\mu_a\}_{a\,\in\,\mathrm{ob}\,J}$ is universal in the sense that for every family $\{\lambda_a : \mathbb{A} \longrightarrow \underline{\mathrm{Hom}}(S_\#(J(a,j)),\, \mathbb{K}_a) \mid a \in \mathrm{ob}\ J\}$ of chain maps ensure the commutativity of the above diagram, there is a unique chain map $h : \mathbb{A} \longrightarrow C^*_J(J(\_,j);\, \mathbb{K})$ such that $\lambda_a = \mu_a \circ h$, $a \in \mathrm{ob}\ J$.

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "\lambda_a"'] \arrow[dr, dotted, "h"] \arrow[ddr, "\lambda_b"'] & & {} \arrow[dr] & \\
 & C^*_J(J(_,j);\, \mathbb{K}) \arrow[rr, "\mu_a"] \arrow[d, "\mu_b"'] & & \underline{\mathrm{Hom}}(S_\#(J(a,j)),\, \mathbb{K}_a) \arrow[d, "\phi^j_{ab}"] \\
 & \underline{\mathrm{Hom}}(S_\#(J(b,j)),\, \mathbb{K}_b) \arrow[rr, "\psi^j_{ab}"'] & & \underline{\mathrm{Hom}}(S_\#(J(b,j)\underline{x}J(a,b)),\, \mathbb{K}_a).
\end{tikzcd}$$

Proof. The claim follows from the the general properties of a equalizer diagram in 3.12.2.

3.12.4 Definition. Define a chain map

$$\lambda_a : \underline{\mathrm{Hom}}_Z(\hat{\mathbf{Z}},\, \mathbb{K}_j) \longrightarrow \underline{\mathrm{Hom}}(S_\#(J(a,j)),\, \mathbb{K}_a),\ a \in \mathrm{ob}\ J,$$

by the assignment

$$\bar{g} = \{g_{p,q}:\hat{\mathbb{Z}}_p \longrightarrow \kappa_q\} \longrightarrow \lambda_a(\bar{g}),$$

where $\lambda_a(\bar{g})_{p,q}:S_p(J(a,j)) \longrightarrow \kappa_q$ is defined by $\lambda_a(\bar{g})_{p,q}(\sigma) = \check{\kappa}(\sigma)(g_{p,q}(1_p))$.

Notation. We write δ^H for the differential of $\underline{\mathrm{Hom}}(\_,\_)$ (as defined in 3.1.1), δ^K for the differentials of $\kappa\_$, δ^* for the differential of $C^*_J(J(\_,j); \kappa)$, and ∂ for the differential of $S_\#(\_)$.

3.12.5 Lemma. Each λ_a is a cochain map.

Proof. Let $\bar{g} = \{g_{p,q}:\hat{\mathbb{Z}}_p \longrightarrow \kappa j_q\} \in \underline{\mathrm{Hom}_{\mathbb{Z}}}(\hat{\mathbb{Z}}, \kappa j)$. Let σ be a singular p–simplex of $J(a,j)$. Then $\lambda_a\delta^H(\bar{g}) \in \underline{\mathrm{Hom}}(S_\#(J(a,j)), \kappa_a)$, and $\lambda_a(\delta^H(\bar{g}))_{p,q}(\sigma) = \check{\kappa}(\sigma)(\delta^H(\bar{g})_{p,q}(1_p)) = \check{\kappa}(\sigma)(g_{p-1,q}\partial(1_p) + (-1)^{p+q}\delta^K g_{p,q-1}(1_p)) = \check{\kappa}(\sigma)(g_{p-1,q}(\sum_{i=0}^{p}(-1)^i 1_{p-1})) + (-1)^{p+q}\check{\kappa}(\sigma)(\delta^K g_{p,q-1}(1_p))$

$$= \sum_{i=0}^{p}(1)^i\check{\kappa}(\sigma)(g_{p-1,q}(1_{p-1})) + (-1)^{p+q}\check{\kappa}(\sigma)\delta^K(g_{p,q-1}(1_p)).$$

Moreover $\delta^H(\lambda_a(\bar{g}))_{p,q}(\sigma) = \lambda_a(\bar{g})_{p-1,q}\partial(\sigma) + (-1)^{p+q}\,\delta^K\lambda_a(\bar{g})_{p,q-1}(\sigma) =$

$$\lambda_a(\bar{g})_{p-1,q}(\sum_{i=0}^{p}(-1)^i\sigma^{(i)}) + (-1)^{p+q}\delta^K\check{\kappa}(\sigma)(g_{p,q-1}(1_p)) =$$

$$\sum_{i=0}^{p}(-1)^i\check{\kappa}(\sigma^{(i)})(g_{p-1,q}(1_{p-1})) + (-1)^{p+q}\,\delta^K\check{\kappa}(\sigma)(g_{p,q-1}(1_p)).$$

Since $\mathrm{image}(\sigma^{(i)}) \subseteq \mathrm{image}(\sigma)$ and $\check{\kappa}(\sigma)$ is a cochain map, $\check{\kappa}(\sigma^{(i)}) = \check{\kappa}(\sigma)$ and $\check{\kappa}(\sigma)\delta^K = \delta^K\check{\kappa}(\sigma)$. Therefore $\lambda_a(\delta^H(\bar{g}))_{p,q}(\sigma) = \delta^H(\lambda_a(\bar{g}))_{p,q}(\sigma)$ as required.

3.12.6 Lemma. Let $a,b \in \operatorname{ob} \mathcal{J}$. The following diagram commutes:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathbb{Z}}(\hat{\mathbb{Z}}, \kappa_j) \arrow[r, "\lambda_a"] \arrow[d, "\lambda_b"'] & \underline{\mathrm{Hom}}(S_{\#}(\mathcal{J}(a,j)), \kappa_a) \arrow[d, "\phi^j_{ab}"] \\
\underline{\mathrm{Hom}}(S_{\#}(\mathcal{J}(b,j)), \kappa_b) \arrow[r, "\psi^j_{ab}"'] & \underline{\mathrm{Hom}}(S_{\#}(\mathcal{J}(b,j)\underline{x}\mathcal{J}(a,b)), \kappa_a)
\end{tikzcd}$$

Proof. Let $\bar{g} = \{g_{p,q}:\hat{\mathbb{Z}}_p \longrightarrow \kappa_{j_q}\} \in \underline{\mathrm{Hom}}_{\mathbb{Z}}(\hat{\mathbb{Z}}, \kappa_j)$. Let (σ,η) be a singular p–simplex of $\mathcal{J}(b,j)\underline{x}\mathcal{J}(a,b)$. Note that $\widetilde{\mathcal{J}(\_,j)}_{ab\#}(\sigma,\eta) = m_{\mathcal{J}\#}(\sigma,\eta) = \sigma \circ \eta$, where $m_{\mathcal{J}}:\mathcal{J}(b,j)\underline{x}\mathcal{J}(a,b) \longrightarrow \mathcal{J}(a,j)$ is the composition of morphisms of $\mathcal{J}$. Then $\phi^j_{ab}(\lambda_a(\bar{g}))_{p,q}(\sigma,\eta) = \phi^j_{ab}(\lambda_a(\bar{g}))_{p,q}(\sigma,\eta) = \lambda_a(\bar{g})(\widetilde{\mathcal{J}(\_,j)}_{ab\#}(\sigma,\eta)) = \lambda_a(\bar{g})(\sigma \circ \eta) = \check{\kappa}(\sigma \circ \eta)(g_{p,q}(1_p))$ and $\psi^j_{ab}(\lambda_b(\bar{g}))_{p,q}(\sigma,\eta) = \check{\kappa}(\eta)(\lambda_b(\bar{g})_{p,q}(\sigma)) = \check{\kappa}(\eta)(\check{\kappa}(\sigma)(g_{p,q}(1_p)))$. By the same method as in the proof of 3.11.6 we can show that $\check{\kappa}(\sigma \circ \eta) = \check{\kappa}(\eta)\circ\check{\kappa}(\sigma)$ ($\check{\kappa}$ is a "contravariant functor") and therefore $\phi^j_{ab}(\lambda_a(\bar{g}))_{p,q}(\sigma,\eta) = \psi^j_{ab}(\lambda_b(\bar{g}))_{p,q}(\sigma,\eta)$ as required.

From 3.12.3 and Lemma 3.12.6, we have

3.12.7 Corollary. There is a unique chain map $h:\underline{\mathrm{Hom}}_{\mathbb{Z}}(\hat{\mathbb{Z}}, \kappa_j) \longrightarrow C^*_{\mathcal{J}}(\mathcal{J}(\_,j);\ \kappa)$ such that $\lambda_a = \mu_a \circ h$, $a \in \operatorname{ob} \mathcal{J}$.

3.12.8 Definition. Here we shall define the inverse of h. Define $f:C^*_J(J(\_,j);\ \kappa) \longrightarrow \underline{\mathrm{Hom}}_Z(\hat{Z},\ \kappa j)$ by the assignment $v \longrightarrow f(v)$, where $f(v)_{p,q}:\hat{Z}_p \longrightarrow \kappa j_q$ is defined by $f(v)_{p,q}(\mathbf{1}_p) = \mu_j(v)_{p,q}(\omega(\mathbf{1}_p))$.

3.12.9 Lemma. f is a chain map.

Proof. Let $v \in C^*_J(J(\_,j);\ \kappa)$. Then $\delta^H(f(v))_{p,q}(\mathbf{1}_p) = f_{p-1,q}(v)\partial(\mathbf{1}_p) + (-1)^{p+q}\delta^K f_{p,q-1}(v)(\mathbf{1}_p) = \sum_{i=0}^{n}(-1)^i f_{p-1,q}(v)(\mathbf{1}_{n-1}) + (-1)^{p+q}\delta^K f_{p,q-1}(v)(\mathbf{1}_p) =$

$\sum_{i=0}^{n}(-1)^i\mu_j(v)_{p-1,q}(\omega(\mathbf{1}_{n-1})) + (-1)^{p+q}\delta^K\mu_j(v)_{p,q-1}(\omega(\mathbf{1}_p))$, and

$f(\delta^*(v))_{p,q}(\mathbf{1}_p) = \mu_j(\delta^*(v))_{p,q}(\omega(\mathbf{1}_p)) = \delta^H(\mu_j(v))_{p,q}(\omega(\mathbf{1}_p)) =$

$\mu_j(v)_{p-1,q}\partial(\omega(\mathbf{1}_p)) + (-1)^{p+q}\ \delta^K\mu_j(v)_{p,q-1}(\omega(\mathbf{1}_p)) =$

$\sum_{i=0}^{n}(-1)^i\mu_j(v)_{p-1,q}(\omega(\mathbf{1}_{p-1})) + (-1)^{p+q}\delta^K\mu_j(v)_{p,q-1}(\omega(\mathbf{1}_p))$. Therefore $\delta^H(f(v))_{p,q}(\mathbf{1}_p) = f(\delta^*(v))_{p,q}(\mathbf{1}_p)$.

3.12.10 Proof of the Proposition. We shall show that the chain map

$$h:\underline{\mathrm{Hom}}_Z(\hat{Z},\ \kappa j) \longrightarrow C^*_J(J(\_,j);\ \kappa)$$

is an isomorphism with the inverse f.

Let $\bar{g} = \{g_{p,q}:\hat{Z}_p \longrightarrow \kappa j_q\} \in \underline{\mathrm{Hom}}_Z(\hat{Z},\ \kappa j)$. Recall that $\lambda_j = \mu_j \circ h$. Then $f(h(\bar{g}))_{p,q}(\mathbf{1}_p) = \mu_j(h(\bar{g}))_{p,q}(\omega(\mathbf{1}_p)) = \lambda_j(\bar{g})_{p,q}(\omega(\mathbf{1}_p)) = \check{\kappa}(\omega(\mathbf{1}_p))(g_{p,q}(\mathbf{1}_p)) = g_{p,q}(\mathbf{1}_p)$ since $\check{\kappa}(\omega(\mathbf{1}_p)) = \kappa(id_j) = 1$; that is, $f \circ h = 1$.

To show that $h \circ f = 1$ we first show that $\mu_a \circ (h \circ f) = \mu_a \circ 1$, $a \in \mathrm{ob}\ J$, and we use the universal property of the family $\{\mu_a\}$, described in 3.12.3, to conclude $h \circ f = 1$. Let $v \in C^*_J(J(\_,j);\ \kappa)$ and $a \in$ ob J. Then $\mu_a \circ h \circ f = \lambda_a \circ f$ and thus $\mu_a(h\circ f(v))_{p,q}(\sigma) = \lambda_a(f(v))_{p,q}(\sigma) = \check{\kappa}(\sigma)(f(v)_{p,q}(1_p)) = \check{\kappa}(\sigma)(\mu_j(v)_{p,q}(\omega(1_p)))$

Immediately from Proposition 2, we have the following corollary.

3.12.11 Corollary.

(1) $H^*_J(J(\_,j);\ \kappa) \cong H^*(\kappa j)$.

(2) If the coefficient chain complex κj is concentrated in zero dimension and have a module K as a zero grading, then $H^*_J(J(\_,j);\ \kappa) \cong K$.

3.12.12 Theorem (DIMENSION). If the coefficient system κ takes values in Ab, then the singular J-cohomology theory satisfies the dimension axiom.

3.13 Additivity

3.13.1 Assume $(X, A) \cong \coprod_i (X_i, A_i)$ in J-CGV. Denote by $u_i:(X_i, A_i) \longrightarrow (X, A)$ the inclusions. Let $a,b \in$ ob J. Then $\sum_i u_{i a\#}: \sum_i S_\#(X_{ia}, A_{ia}) \cong S_\#(X_a, A_a)$ and $\sum_i (u_{ib} \underline{x} 1_{J(a,b)})_\#: \sum_i S_\#((X_{ib}, A_{ib})\underline{x}J(a,b)) \cong S_\#((X_b, A_b)\underline{x}J(a,b))$ with all isomorphisms natural.

3.13.2 Immediately from the construction of the singular J homology we have the following commutative diagram

$$\begin{array}{ccc}
\sum\limits_{a,b} S_{\#}((X_b, A_b) \underline{x} J(a,b)) \otimes_Z M_a & \overset{\alpha^{X,A}}{\underset{\beta^{X,A}}{\rightrightarrows}} & \sum\limits_{c} S_{\#}(X_c) \otimes_Z M_c \\
\Big\uparrow \hat{u}_1 = \sum\limits_i \hat{u}_1^{\,i} & & \Big\uparrow \hat{u}_2 = \sum\limits_i \hat{u}_2^{\,i} \\
\sum\limits_i \sum\limits_{a,b} S_{\#}((X_b, A_b) \underline{x} J(a,b)) \otimes_Z M_a & \overset{\Sigma\alpha^{X,A}}{\underset{\Sigma\beta^{X,A}}{\rightrightarrows}} & \sum\limits_i \sum\limits_{c} S_{\#}(X_c) \otimes_Z M_c
\end{array} ,$$

where clearly $\hat{u}_1$ and $\hat{u}_2$ are isomorphisms. Therefore $\sum\limits_i u_{\#}^{i}$ induced by $u = \coprod u_i$ gives an isomorphism of the direct sum of chain complexes $\sum\limits_c C_*(X_i, A_i; M)$ with $C_*(X,A; M)$.

3.13.4 Theorem (ADDITIVITY). The homomorphism $\sum\limits_i u_*^{i}: \sum\limits_c H_*^J(X_i, A_i; M) \longrightarrow H_*^J(X,A; M)$ is an isomorphism.

Proof. This follows immediately from the fact that $\sum\limits_i u_{\#}^{i}$ is an isomorphism (see 3.13.3).

3.13.5 Similarly, from the definition of the J–cohomology we have

$$\begin{array}{ccc}
\prod_a \underline{\mathrm{Hom}}(S_\#(X_a, A_a), \mathbb{K}_a) & \overset{\phi^{X,A}}{\underset{\psi^{X,A}}{\rightrightarrows}} & \prod_{b,c} \underline{\mathrm{Hom}}(S_\#((X_c, A_c) \underline{\times} \mathcal{J}(b,c)), \mathbb{K}_b) \\
\check{u}_1 = \prod_i \check{u}_{i\,1} \Big\downarrow & & \Big\downarrow \check{u}_2 = \prod_i \check{u}_{i\,2} \\
\prod_i \prod_a \underline{\mathrm{Hom}}(S_\#(X_{i\,a}, A_{i\,a}), \mathbb{K}_a) & \overset{\prod\phi^{X,A}}{\underset{\prod\psi^{X,A}}{\rightrightarrows}} & \prod_i \prod_{b,c} \underline{\mathrm{Hom}}(S_\#((X_{i\,c}, A_{i\,c}) \underline{\times} \mathcal{J}(b,c)), \mathbb{K}_b),
\end{array}$$

where clearly $\prod_i \check{u}_1$ and $\prod_i \check{u}_2$ are isomorphisms. Therefore $\prod_i u_i^\#$ induced by $u = \coprod_i u$ gives an isomorphism of the cochain complex $C^*_{\mathcal{J}}(X,A;\ \mathbb{K})$ with the product of cochain complexes $\prod_i C^*_{\mathcal{J}}(X_i, A_i;\ \mathbb{K})$.

3.13.6 Theorem (WEDGE - ADDITIVITY). The homomorphism $\prod_i u_i^*: H^*_{\mathcal{J}}(X,A;\ \mathbb{K}) \longrightarrow \prod_i H^*_{\mathcal{J}}(X_i, A_i;\ \mathbb{K})$ is an isomorphism.

Proof. This follows directly from the fact that $\prod_i u_i^\#$ is an isomorphism (see 3.13.5).

3.14 Specialization to Ends and Coends: The Case When $\mathcal{J}$ is a Discrete Category

A category $\mathcal{J}$ is called **discrete** if for all $a,b \in \mathrm{ob}\ \mathcal{J}$ each space $\mathcal{J}(a,b)$ has the discrete topology. Assume that coefficient systems on $\mathcal{J}$ are taking values in ${}_R\mathrm{Mod}$. We shall show first that in the case of a discrete J the singular homology and cohomology of any diagram are given by the classical coend and end construction (see [MacLane 2]). Classical coends and ends are used in [Piacenza 1] to define homology and cohomology for functors from a small category to simplicial sets. We shall show that for a discrete J the singular J-(co)homology defined in this thesis is isomorphic to the J-(co)homology as defined in [Piacenza 1]. Note that for a discrete group G an analogous construction is presented in [tom Dieck], Section II.9, to define equivariant homology theory.

3.14.1 Let X and Y be two topological spaces. Assume that Y is a discrete topological space. Note that always $S_n(X\underline{x}Y) \cong S_n(X) \otimes_{\mathbf{Z}} S_n(Y)$, n = 0,1,2,.... Since Y is discrete, the image image(η) of any singular n-simplex $\eta:\Delta_n \longrightarrow Y$ is a point of Y. For a point y of Y denote by $[y]^n:\Delta_n \longrightarrow Y$ the unique singular n-simplex of Y with image($[y]^n$) = $\{y\}$. Note that each $S_n(Y)$ is canonically isomorphic to $\mathcal{F}(Y)$, the free $\mathbf{Z}$-module generated by the set Y (compare 3.2.4); obviously, the canonical isomorphism $\mathcal{F}(Y) \cong S_n(Y)$ is induced by the correspondence $y \longrightarrow [y]^n$. Therefore, we have $S_n(X\underline{x}Y) \cong S_n(X) \otimes_{\mathbf{Z}} \mathcal{F}(Y)$, n = 0,1,2,....

3.14.2 Let $(X,A):J^{op} \longrightarrow \mathrm{CGV}$, $\mathbb{M}:J \longrightarrow {}_R\mathrm{Mod}$, $\mathbb{K}:J^{op} \longrightarrow {}_R\mathrm{Mod}$. Then the singular functor S induces functors $S_n(X,A):J^{op} \longrightarrow {}_{\mathbf{Z}}\mathrm{Mod}$, n = 0,1,2,..., defined by $S_n(X,A)(a) = S_n(X_a,A_a)$ for $a \in \mathrm{ob}\ J$, and $S_n(X,A)(f) = S_n(X(f)):S_n(X_b,A_b) \longrightarrow S_n(X_a,A_a)$ for $f \in J(a,b)$.

3.14.3 Theorem. The chain complex $C^{J}_{*}(X,A;\ \mathbb{M})$ is isomorphic to a coend chain complex $S_*(X,A) \otimes_J \mathbb{M},\ \partial_* \otimes_J 1$.

Proof. Recall that $C^{J}_{n}(X,A;\ \mathbb{M})$ is defined by the following coequalizer diagram

$$\sum_{a,b} S_n\big((X_b,A_b)\underline{x}J(a,b)\big) \otimes_Z \mathbb{M}_a \underset{\beta^{X,A}}{\overset{\alpha^{X,A}}{\rightrightarrows}} \sum_{c} S_n(X_c,A_c) \otimes_Z \mathbb{M}_c,$$

where $[(\sigma,\eta)]\otimes m \xrightarrow{\alpha^{X,A}} \tilde{X}_{ab\#}(\sigma,\eta)\otimes m$ and $([\sigma,\eta])\otimes m \xrightarrow{\beta^{X,A}} [\sigma]\otimes\check{\mathbb{M}}(\eta)(m)$ (see 3.3.4, 3.3.6, and 3.3.10). Note that if $\eta = [f]^n$, then $\tilde{X}_{ab\#}([\sigma,\eta]) = S_n(X(f))([\sigma]) = S_n(X,A)(f)([\sigma])$ and $\check{\mathbb{M}}(\eta) = \mathbb{M}(f)$. Isomorphisms $S_n\big((X_b,A_b)\underline{x}J(a,b)\big) \cong S_n\big((X_b,A_b)\big) \otimes \mathcal{F}\big(J(a,b)\big)$, $a,b \in \text{ob } J$, yield an isomorphism:

$$\sum_{a,b} S_n\big((X_b,A_b)\underline{x}J(a,b)\big) \otimes \mathbb{M}_a \xrightarrow{\cong} \sum_{a,b} S_n(X_b,A_b) \otimes \mathcal{F}\big(J(a,b)\big) \otimes \mathbb{M}_a.$$

Moreover, we have the following isomorphism of R–modules:

$$\sum_{a,b\in \text{ ob } J} S_n(X_b,A_b) \otimes \mathcal{F}\big(J(a,b)\big) \otimes \mathbb{M}_a \xrightarrow{\cong} \sum_{f\in \text{ Morph}(J)} S_n(X_{\mathrm{cod}(f)},A_{\mathrm{cod}(f)}) \otimes \mathbb{M}_{\mathrm{dom}(f)}.$$

Note that under these two isomorphisms the coequalizer diagram which defines $C^{J}_{n}(X,A;\ \mathbb{M})$ can be rewritten as the classical coend diagram for functors $S_n(X,A)$ and $\mathbb{M}$ (see [tom Dieck] I.11.6)

$$\sum_{f\in \text{ Morph}(J)} S_n(X_{\mathrm{cod}(f)},A_{\mathrm{cod}(f)}) \otimes \mathbb{M}_{\mathrm{dom}(f)} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \sum_{c\in \text{ ob } J} S_n(X_c,A_c) \otimes \mathbb{M}_c,$$

where $[\sigma]\otimes m \xrightarrow{\alpha} S_n(X,A)(f)([\sigma])\otimes m$ and $[\sigma]\otimes m \xrightarrow{\beta} [\sigma]\otimes\mathbb{M}(f)(m)$ for $[\sigma]\otimes m \in S_n(X_{\mathrm{cod}(f)},A_{\mathrm{cod}(f)}) \otimes \mathbb{M}_{\mathrm{dom}(f)}$. In a similar way we obtain that $\partial_*\otimes_J 1$ is the induced boundary operator.

3.14.4 Theorem. The cochain complex $C^*_{\mathcal{J}}(X,A;\ \kappa)$ is isomorphic to an end $\underline{\mathrm{Hom}}_{\mathcal{J}}(S_*(X,A),\ \kappa),\ \underline{\mathrm{Hom}}_{\mathcal{J}}(\partial_*,\ 1_\kappa)$.

Proof. $C^n_{\mathcal{J}}(X,A;\ \kappa)$ is defined by the following equalizer diagram

$$\prod_{a} \mathrm{Hom}(S_n(X_a,A_a),\ \kappa_a) \underset{\psi^{X,A}}{\overset{\phi^{X,A}}{\rightrightarrows}} \prod_{b,c} \mathrm{Hom}(S_n((X_c,A_c)\underline{\times}\mathcal{J}(b,c)),\ \kappa_b),$$

where $\phi^{X,A}(\{f_a\}_a)_{b,c} = f_b\circ(\tilde{X}_{bc},\tilde{A}_{bc})_\#: S_n((X_c,A_c)\underline{\times}\mathcal{J}(b,c)) \longrightarrow \kappa_b$ and $\psi^{X,A}(\{f_a\}_a)_{b,c}([\sigma,\eta]) = \check{\kappa}(\eta)(f_c([\sigma]))$ for $f_a: S_n(X_a,A_a) \longrightarrow \kappa_a$ (see 3.4.1, 3.4.4, and 3.4.9). Note that if $\eta = [f]^n$, then $\tilde{X}_{ab\#}([\sigma,\eta]) = S_n(X(f))([\sigma])$ $S_n(X,A)(f)([\sigma])$ and $\check{\kappa}(\eta) = \kappa(f)$. Isomorphisms $S_n((X_b,A_b)\underline{\times}\mathcal{J}(a,b)) \cong S_n((X_b,A_b)) \otimes \mathcal{F}(\mathcal{J}(a,b))$, $a,b \in \mathrm{ob}\ \mathcal{J}$, yield an isomorphism:

$$\prod_{b,c} \mathrm{Hom}(S_n((X_c,A_c)\underline{\times}\mathcal{J}(b,c)),\ \kappa_b) \xrightarrow{\cong} \prod_{b,c} \mathrm{Hom}(S_n(X_c,A_c)\otimes\mathcal{F}(\mathcal{J}(b,c)),\ \kappa_b).$$

Moreover, there is the following isomorphism of R-modules:

$$\prod_{b,c} \mathrm{Hom}(S_n(X_c,A_c)\otimes\mathcal{F}(\mathcal{J}(b,c)),\ \kappa_b) \xrightarrow{\cong} \prod_{f\in \mathrm{Morph}(\mathcal{J})} \mathrm{Hom}(S_n(X_{\mathrm{cod}(f)},A_{\mathrm{cod}(f)}),\ \kappa_{\mathrm{dom}(f)}).$$

Note that under these two isomorphisms the equalizer diagram which defines $C^n_{\mathcal{J}}(X,A;\ \kappa)$ can be rewritten as the classical end diagram for functors $S_n(X,A)$ and κ ([MacLane 2]) p. 109)

$$\prod_{a\in \mathrm{ob}\ \mathcal{J}} \mathrm{Hom}(S_n(X_a,A_a),\ \kappa_a) \underset{\psi}{\overset{\phi}{\rightrightarrows}} \prod_{f\in \mathrm{Morph}(\mathcal{J})} \mathrm{Hom}(S_n(X_{\mathrm{cod}(f)},A_{\mathrm{cod}(f)}),\ \kappa_{\mathrm{dom}(f)}),$$

where $\phi(\{f_a\}_a)_f([\sigma]) = f_{\mathrm{dom}(f)}\circ S_n(X,A)(f)([\sigma])$ and $\psi(\{f_a\}_a)_f([\sigma]) = \kappa(f)(f_{\mathrm{cod}(f)}([\sigma]))$ for $[\sigma] \in S_n(X_{\mathrm{cod}(f)},A_{\mathrm{cod}(f)})$. In a similar way we obtain that $\underline{\mathrm{Hom}}_{\mathcal{J}}(\partial_*,\ 1_\kappa)$ is the induced coboundary operator.

3.14.5 The Singular (Co)homology for Diagrams via Simplicial Sets ([Piacenza 1]). If J is a discrete category and $X:J^{op} \longrightarrow \mathbf{CGV}$, then an application of the singular functor gives us a functor $\mathcal{S}X$ from J^{op} to the category of simplicial sets $\mathbf{SS}$. Let $L:\mathbf{SS} \longrightarrow \mathbf{Compl}$ be the free chain complex functor as in [May 2] p. 5. Then $L(\mathcal{S}X) = S_*(X)$ with $S_n(X)$ defined in 3.14.2. A coefficient system on J is a (co)functor from J to $\mathbf{Ab}$. For a special case of $\mathcal{S}X$ and a coefficient system $\mathcal{M}$ ($\mathcal{K}$), (co)homology groups are given by $H_*(\mathcal{S}X; \mathcal{M}) \stackrel{\text{def}}{=} H_*(L(\mathcal{S}X)\otimes_J \mathcal{M}) = H_*(S_*(X)\otimes_J \mathcal{M})$ and $H^*(\mathcal{S}X; \mathcal{K}) \stackrel{\text{def}}{=} H_*(\underline{\mathrm{Hom}}_J(L(\mathcal{S}X), \mathcal{K})) = H^*(\underline{\mathrm{Hom}}_J(S_*(X), \mathcal{K}))$ ([Piacenza 1], Section II). Therefore, from 3.14.4 and 3.14.5 we have the following theorem.

3.14.6 Theorem. If J is a discrete category then singular J-(co)homology for a diagram X of topological spaces specializes to (co)homology of the induced diagram $\mathcal{S}X$ of simplicial sets.

3.14.7 The equivariant (co)homology for G-spaces as presented in [tom Dieck], pp. 161-163. Let G be a finite group. Denote by $\hat{G}$ the category of finite G-sets and G-maps between them. A pair of G-spaces (X,A) induces diagrams $\underline{X}:\hat{G} \longrightarrow \mathbf{CGV}$, $\underline{X}(a) = G\text{-}\mathrm{Top}(a, X)$ (a space of G-maps from a to X) and $\underline{A}(a) = G\text{-}\mathrm{Top}(a, A)$, $a \in \mathrm{ob}\, \hat{G}$. A coefficient system on $\hat{G}$ is prescribed as a Mackey functor $\mathcal{M} = (\mathcal{M}^*, \mathcal{M}_*)$, a pair of a contravariant functor $\mathcal{M}^*:\hat{G}^{op} \longrightarrow {}_{\mathbb{Z}}\mathrm{Mod}$ and a covariant functor $\mathcal{M}_*:\hat{G} \longrightarrow {}_{\mathbb{Z}}\mathrm{Mod}$; the functors are assumed to coincide on objects and to obey some additional properties which are irrelevant for us (see [tom Dieck] II.9.2–9.3). Then the equivariant homology $H_*(X,A; \mathcal{M}_*)$ of a pair (X,A) of G-spaces is taken to be a homology of the chain complex $S_{\#}(\mathrm{Hom}_G(\_,X), \mathrm{Hom}_G(\_,A)) \otimes_{\hat{G}} M_*$ (see [tom Dieck] II.9.4). From Theorem 3.14.3 these homology groups are isomorphic to $H^{J}_{*}(\underline{X},\underline{A};\, M_*)$. The equivariant cohomology groups $H^*(X,A;\, M^*)$ are defined via the cochain complex $\mathrm{Hom}_{\hat{G}}(S_{\#}(\mathrm{Hom}_G(\_,X),\mathrm{Hom}_G(\_,A)),\, M^*)$ and are isomorphic to $H^*_J(\underline{X},\underline{A};M^*)$ by Theorem 3.14.4.

CHAPTER 4

SHEAF *J*-COHOMOLOGY

In this chapter we shall define a sheaf cohomology of diagrams of topological spaces and establish its relation to singular *J*-cohomology. In the first section we shall define a natural presheaf associated with given diagram of topological spaces and a continuous contravariant coefficient system. This presheaf allows us to use classical sheaf cohomology to define a "sheaf" cohomology for a diagram X. In the second part of the chapter we prove, with the imposition of additional assumptions, the existence of an isomorphism between singular and sheaf cohomology for diagrams. We shall assume that the reader is familiar with the development of sheaf theory as is found in the first two chapters of Bredon's *Sheaf Theory*. In particular, we will assume familiarity with notions of a presheaf (p. 2), a sheaf (p. 3), an induced sheaf (p. 4), a sheaf cohomology (p. 28), and the relative sheaf cohomology (p. 57).

4.1 Sheaf *J*-Cohomology

4.1.1 For a given diagram X and a coefficient system, a presheaf of modules is defined on the topological space $\mathrm{colim}_J X$ following that in **[Piacenza 2]**. Using this presheaf an application of the classical sheaf theory shall instantaneously allow us to define a sheaf cohomology for diagrams of topological spaces and to establish its properties.

4.1.2 Definition ([Piacenza 2], p. 818). Let $\kappa{:}J^{op} \longrightarrow \mathbf{Mod}_R$ (or **Ab**) be a continuous coefficient system on J. Let $X{:}J^{op} \longrightarrow \mathbf{CGV}$ be a diagram on J. Recall that for any open $\mathfrak{U} \subseteq X/J$, there is a subdiagram $\check{\mathfrak{U}}$ of X (see 1.4.9). Note that κ can be treated as a functor with values in **CGV**. For X define a presheaf κX of R-modules (resp. abelian groups) on the topological space X/J by

$$\kappa X(\mathfrak{U}) = J\text{-}\mathbf{CGV}(\check{\mathfrak{U}}, \kappa)$$

(assume it has the discrete topology defined on it) for an open set $\mathfrak{U} \subseteq X/J$, with restrictions $r_{U,V} = J\text{-}\mathbf{CGV}(i_V^U, 1_K){:}\kappa X(V) \longrightarrow \kappa X(\mathfrak{U})$ for $i_V^U{:}\mathfrak{U} \subseteq V$. Denote by $\mathcal{K}X$ the sheaf generated by κX.

4.1.3 Let Υ be a topological space and A be a presheaf on Υ. Recall the following (S1) and (S2) conditions from [Bredon 1], p. 5.

(S1): If $\mathfrak{U} = \bigcup_k \mathfrak{U}_k$, with $\mathfrak{U}_k$ open in Υ, and $s,t \in A(\mathfrak{U})$ are such that $s|\mathfrak{U}_k = t|\mathfrak{U}_k$ for all k, then $s = t$.

(S2): Let $\{\mathfrak{U}_k\}$ be a collection of open sets in Υ and let $\mathfrak{U} = \bigcup \mathfrak{U}_k$. If $s_k \in A(\mathfrak{U}_k)$ are given such that $s_k|\mathfrak{U}_k \cap \mathfrak{U}_n = s_n|\mathfrak{U}_k \cap \mathfrak{U}_n$ for all k,n then there exists an element $s \in A(\mathfrak{U})$ with $s|\mathfrak{U}_k = s_k$ for all k.

4.1.4 Lemma. κX satisfies S1 and S2 conditions.

Proof. Recall that a natural transformation from a diagram X to Y is a family of continuous functions $\{\lambda_a{:}X_a \longrightarrow Y_a |\ a \in \text{ob } J\}$ satisfying the naturality condition. Observe that both conditions (S1) and (S2) (4.1.3), when $a \in \mathrm{ob}\, J$ is fixed, becomes assertions about (continuous) functions (defined on open coverings) and thus the claim is easily seen to be true.

4.1.5 Definition ([Piacenza 2], 4.2). Define an absolute sheaf J-cohomology of X with coefficients in κ by

$$ {}_{\Delta h}H^*_J(X; \kappa) = H^*(X/J; \mathcal{K}X) $$

where the right side is the sheaf cohomology of the topological space X/J with coefficients in the sheaf $\mathcal{K}X$, and supports being closed sets, as in [**Bredon 1**], Definition II.2.1, p.28.

4.1.6 Definition ([Piacenza 2], 4.4). A special pair $(X,\mathbf{A})$ is called a sheaf **acceptable** pair if for each coefficient system κ on J the sheaf $\mathcal{K}\check{\mathbf{A}}$ on $\mathbf{A}$, where $\mathcal{K}\check{\mathbf{A}}$ is the sheafication of $\kappa\check{\mathbf{A}}$, is the restriction of the sheaf $\mathcal{K}X$ to the subspace $\mathbf{A}$. Note that if $(X,\mathbf{A})$ is a J-NR pair, or if X is locally J-NR, or if $\mathbf{A}$ is open, then $(X,\mathbf{A})$ is acceptable. Denote by $\mathcal{A}d_{\Delta h}$ the full subcategory of acceptable pairs in J-CGV^2 and observe that $\mathcal{A}d_{\Delta h}$ is an admissible category as defined in 2.1.

4.1.7 Definition ([Piacenza 2], p. 819). Let $(X,\mathbf{A})$ be a acceptable pair. Define relative J-cohomology of $(X,\mathbf{A})$ with coefficients in κ by

$$ {}_{\Delta h}H^*_J(X,\mathbf{A}; \kappa) = H^*(X/J,\mathbf{A}; \mathcal{K}X) $$

where the right side is the sheaf cohomology of the pair $(X/J,\mathbf{A})$ with coefficients in the sheaf $\mathcal{K}X$, and supports being closed sets (as in [**Bredon**

1], Section II.12, p.57).

4.1.8 Theorem. The sheaf J–cohomology satisfies the Homotopy Axiom, the Exactness Axiom, the Excision Axiom, and the Dimension Axiom.

Proof. The claim follows from well–known properties of classical sheaf cohomology. The Homotopy Axiom follows easily from [Bredon 1], Corollary II.11.3, p. 56. The Exactness Axiom is established in [Bredon 1], II.12(2), p. 58, after observing that $\mathcal{K}X|\mathbf{A} = \mathcal{K}\check{\mathbf{A}}$ yields $H^*(\mathbf{A};\ \mathcal{K}X|\mathbf{A}) = H^*(\mathbf{A};\ \mathcal{K}\check{\mathbf{A}}) = {}_{\mathit{sh}}H^*_J(\check{\mathbf{A}};\ \kappa)$. The Excision Axiom is proved in a more general form in [Bredon 1], Theorem II.12.4, p. 61. To prove the Dimension Axiom note that $J(\_,j)/J = \mathrm{colim}_J J(\_,j) = *$ (a point) and $\kappa J(\_,j) \cong \kappa(j)$. Therefore ${}_{\mathit{sh}}H^n_J(J(\_,j);\ \kappa) = H^n(*;\ \kappa(j)) \cong \begin{cases} \kappa(j) & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$ as required.

4.2 Comparison Theorem

In this section we shall compare singular J–cohomology and the sheaf J–cohomology. The main theorem 4.2.20 extends a similar result, Theorem 3.11 from [Piacenza 1], to the case of singular J–cohomology.

4.2.1 Let X be a diagram. Recall that the singular J–cochain complex $C^*_J(X;\ \kappa) = \mathrm{kernel}(\Psi^X)$ is defined with the help of the following equalizer diagram

$$C^*_J(X;\,\kappa) \longrightarrow \prod_a \underline{\mathrm{Hom}}(S_\#(X_a),\,\kappa_a) \underset{\psi^X}{\overset{\phi^X}{\rightrightarrows}} \prod_{b,c} \underline{\mathrm{Hom}}(S_\#(X_c \underline{x} J(b,c)),\,\kappa_b)$$

where $\Psi^X = \phi^X - \psi^X$. Since κ is a module only, $[\underline{\mathrm{Hom}}(S_\#(X_a),\,\kappa_a)]_n = \mathrm{Hom}_Z(S_n(X_a),\,\kappa_a) = S^n(X_a;\,\kappa_a)$, called traditionally the complex of n–cochains of X_a with values in the R–module κ_a. Similarly, $[\underline{\mathrm{Hom}}(S_\#(X_c \underline{x} J(b,c)),\,\kappa_b)]_n = S^n(X_c \underline{x} J(b,c);\,\kappa_b)$. Therefore $C^*_J(X;\,\kappa)$ is determined by the diagram

$$C^*_J(X;\,\kappa) \longrightarrow \prod_a S^*(X_a;\,\kappa_a) \underset{\psi^X}{\overset{\phi^X}{\rightrightarrows}} \prod_{b,c} S^*(X_c \underline{x} J(b,c);\,\kappa_b).$$

4.2.2 Definition. Let $n \geq 0$. For X we denote by $C^n = C^n(X;\,\kappa)$ a presheaf on X/J of R–modules defined by $C^n(U) = C^n_J(\check{U};\,\kappa)$, U open in X/J, with restriction homomorphisms $r_{U,V} = (i^U_V)^\#: C^n(V) \longrightarrow C^n(U)$ for the inclusion $i^U_V: U \subseteq V$. Define presheaves A^n and B^n on X/J by

$$A^n(U) = \prod_a S^*(t_a^{-1}(U);\,\kappa_a) \quad \text{and} \quad B^n(U) = \prod_{b,c} S^n(t_c^{-1}(U)\underline{x}J(b,c);\,\kappa_b)$$

for an open subset U of X/J and with obvious restriction homomorphisms; $t_a: X_a \longrightarrow X/J$, $a \in \mathrm{ob}\,J$, is the canonical map from X_a to the colimit X/J. Note that $C^*(U)$ is the equalizer cochain complex in the diagram

$$C^*(U) \longrightarrow A^*(U) \underset{\psi^U}{\overset{\phi^U}{\rightrightarrows}} B^*(U).$$

Therefore C^* is a differential presheaf. Denote by $\mathcal{C}^n = \mathcal{C}^n(X;\,\kappa)$ the sheafication of C^n, and by $\mathcal{C}^* = \mathcal{C}^*(X;\,\kappa)$ the induced differential sheaf.

4.2.3 Lemma. Let $\mathfrak{U}$ be an open set in X/J. Then $C^0(\mathfrak{U}) = J\text{-}\mathbf{Sets}(\check{\mathfrak{U}}, \mathfrak{K})$, that is, $C^0(\mathfrak{U})$ can be viewed as the set of all natural transformations, not necessary continuous, from $\check{\mathfrak{U}}$ to $\mathfrak{K}$.

Proof. Note that $C^0(\mathfrak{U}) \longrightarrow A^0(\mathfrak{U}) \underset{\psi^U}{\overset{\phi^U}{\rightrightarrows}} B^0(\mathfrak{U})$, where

$$A^0(\mathfrak{U}) = \prod_a \mathbf{Sets}(t_a^{-1}(\mathfrak{U}); \mathfrak{K}_a) \quad \text{and} \quad B^0(\mathfrak{U}) = \prod_{b,c} \mathbf{Sets}(t_c^{-1}(\mathfrak{U})\underline{x}J(b,c); \mathfrak{K}_b)$$

Let $f = \{f_a\} \in A^0(\mathfrak{U})$ such that $\phi^U(f) = \psi^U(f)$. Recall that $\phi^U(f)_{ab} = \phi^U_{a,b}(f_a)$ $= f_a \circ (\tilde{X}_{ab}|\check{\mathfrak{U}}):\check{\mathfrak{U}}_b\underline{x}J(a,b) \longrightarrow \check{\mathfrak{U}}_a$ with $\phi^U_{a,b}(f_a)(x,g) = f_a(\tilde{X}_{ab}(x,g))$, and $\psi^U(f)_{ab} = \psi^U_{a,b}(f_b):\check{\mathfrak{U}}_b\underline{x}J(a,b) \longrightarrow \check{\mathfrak{U}}_a$ with $\psi^U_{a,b}(f_b)(x,g) = \mathfrak{K}(g)(f_b(x))$, for $(x,g) \in \check{\mathfrak{U}}_b\underline{x}J(a,b)$. Thus $\phi^U(f) = \psi^U(f)$ yields $f_a(\tilde{X}_{ab}(x,g)) = \mathfrak{K}(g)(f_b(x))$ for $(x,g) \in \check{\mathfrak{U}}_b\underline{x}J(a,b)$, which is equivalent to the commutativity of the following diagram in the category of sets:

$$\begin{array}{ccc} \check{\mathfrak{U}}_b\underline{x}J(a,b) & \xrightarrow{\tilde{X}_{ab}|} & \check{\mathfrak{U}}_a \\ {\scriptstyle f_b\underline{x}1}\big\downarrow & & \big\downarrow{\scriptstyle f_a} \\ \mathfrak{K}_b\underline{x}J(a,b) & \xrightarrow[\tilde{\mathfrak{K}}_{ab}]{} & \mathfrak{K}_a \end{array} .$$

The commutativity of this diagram asserts that the family $f = \{f_a\}$ is a natural transformation from $\mathfrak{U}$ to $\mathfrak{K}$ both viewed as functors into **Sets**.

4.2.4 Lemma.

(1) Each presheaf C^n, $n \geq 0$, satisfies the condition (S2).

(2) The presheaf C^0 satisfies both (S1) and (S2).

Proof. (1) Let $\{\mathcal{U}_\alpha\}$ be a collection of open sets in X/J and let $\mathcal{U} = \bigcup\{\mathcal{U}_\alpha\}$.

Assume that $c_\alpha \in C^n(\mathcal{U}_\alpha)$ are given such that $c_\alpha|\mathcal{U}_\alpha\cap\mathcal{U}_\beta = c_\beta|\mathcal{U}_\alpha\cap\mathcal{U}_\beta$ for all α,β.

Let $c^*_\alpha = \gamma_\alpha(c_\alpha)$ be the image of c_α in $A^n(\mathcal{U}_\alpha)$. Note that c^*_α is in the kernel of Ψ if $\phi^U(c^*_\alpha)_{ab} = \psi^U(c^*_\alpha)_{ab}$ for all a and b, or equivalently $\phi^U_{ab}(c^*_{\alpha_a}) = \psi^U_{ab}(c^*_{\alpha_b})$. The last equality reads $c^*_{\alpha_a}(\tilde{X}_{ab\#}(\sigma,\eta)) = \check{\kappa}(\eta)(c^*_{\alpha_b}(\sigma))$ for a singular n-simplex of $\check{\mathcal{U}}_{\alpha_b}\underline{x}J(a,b)$.

Define $c^* \in A^n(\mathcal{U})$ by $c^*_a(\sigma) = \begin{cases} c^*_{\alpha_a}(\sigma) & \exists\,\alpha \quad \sigma \subseteq \check{\mathcal{U}}_{\alpha_a} \\ 0 & \text{otherwise} \end{cases}$, for $a \in \mathrm{ob}\,J$ and a singular n-simplex σ of $\check{\mathcal{U}}_a$. The cochain c^* is in the kernel of Ψ if $\phi^U(c^*)_{ab} = \psi^U(c^*)_{ab}$ for all a, b or equivalently if $\phi^U_{ab}(c^*_a) = \psi^U_{ab}(c^*_b)$. Then

$$\phi^U_{ab}(c^*_a)(\sigma,\eta) = c^*_a(\tilde{X}_{ab\#}(\sigma,\eta)) = \begin{cases} c^*_{\alpha_a}(\tilde{X}_{ab\#}(\sigma,\eta)) & \exists\,\alpha \quad \tilde{X}_{ab\#}(\sigma,\eta) \subseteq \check{\mathcal{U}}_{\alpha_a} \\ 0 & \text{otherwise} \end{cases}.$$

Note that if $\mathrm{image}(\tilde{X}_{ab\#}(\sigma,\eta)) \subseteq \check{\mathcal{U}}_{\alpha_a} = t_a^{-1}(\mathcal{U}_\alpha)$, then by the properties of the colimit X/J $\mathrm{image}(\sigma) \subseteq \check{\mathcal{U}}_{\alpha_b} = t_b^{-1}(\mathcal{U}_\alpha)$. Therefore

$$\phi^U_{ab}(c^*_a)(\sigma,\eta) = \begin{cases} c^*_{\alpha_a}(\tilde{X}_{ab\#}(\sigma,\eta)) & \exists\,\alpha \quad \sigma \subseteq \check{\mathcal{U}}_{\alpha_a} \\ 0 & \text{otherwise} \end{cases}.$$ Moreover,

$$\psi^U_{ab}(c^*_b)(\sigma,\eta) = \check{\kappa}(\eta)(c^*_b(\sigma)) = \begin{cases} \check{\kappa}(\eta)(c^*_{\alpha_a}(\sigma)) & \exists\,\alpha \quad \sigma \subseteq \check{\mathcal{U}}_{\alpha_a} \\ 0 & \text{otherwise} \end{cases}.$$

Since c^*_α is in the kernel of Ψ, $c^*_{\alpha a}(\tilde{X}_{ab\#}(\sigma,\eta)) = \check{\kappa}(\eta)(c^*_{\alpha b}(\sigma))$, and therefore $\phi^U_{ab}(c^*_a)(\sigma,\eta) = \psi^U_{ab}(c^*_b)(\sigma,\eta)$ as required. (2) From the characterization given in 4.2.3 $C^0(U) = \mathcal{J}\text{-Sets}(\check{U}, \kappa)$. The conditions (S1) and (S2) are obviously satisfied when checking them for each $a \in$ ob $\mathcal{J}$.

4.2.5 Definition. Let θ be the canonical homomorphism from the presheaf C^n to its sheafication $\mathcal{C}^n$ (see [Bredon 1], I.1.5). Denote $C^n_0 = C^n_0(X; \kappa)$, $n \geq 0$, a presheaf defined by

$$C^n_0(U) = \{s \in C^n(U) \mid \theta_U(s) = 0 \text{ in } \mathcal{C}^n(U)\}$$
$$= \{s \in C^n(U) \mid \exists \ \{U_i\} \text{ an open cover of } U,\ s|U_i = 0 \text{ in } C^n(U_i)\}.$$

Denote by $C^*_0 = C^*_0(X; \kappa)$ the induced differential presheaf; C^*_0 is called the subcomplex of locally zero cochains of C^* (see [Spanier], p. 326, Example 9).

4.2.6 Lemma. $H^*(C^*_0(X)) = 0$ and $C^0_0(X) = 0$.

Proof. Let $\mathcal{U}$ be a covering of $X/\mathcal{J}$ by open sets and let $C^*_{\mathcal{J}}(\mathcal{U}; \kappa)$ be the complex of singular $\mathcal{J}$-cochains based on $\mathcal{U}$-small singular $\mathcal{J}$-cochains as defined in 3.10.14. Then the subdivision argument (3.10.24) shows that $i_{\mathcal{U}}: C^*_{\mathcal{J}}(X; \kappa) \longrightarrow C^*_{\mathcal{J}}(\mathcal{U}; \kappa)$ induces an isomorphism in cohomology, and therefore, $H^*(\ker(i_{\mathcal{U}})) = 0$. Observe that $C^*_0(X) = \varinjlim_{\mathcal{U}} \ker(i_{\mathcal{U}})$, so that $H^*(C^*_0(X)) = \varinjlim_{\mathcal{U}} H^*(\ker(i_{\mathcal{U}})) = 0$.

4.2.7 Lemma. Let ϕ be the paracompactifying family of supports on X. The sequence of complexes of modules

$$0 \longrightarrow C_0^*(X) \longrightarrow C_\phi^*(X) \xrightarrow{\theta} \Gamma_\phi \mathcal{E}^* \longrightarrow 0$$

is exact.

Proof. All of the C^n satisfy (S2) and the assertion follows from **[Bredon 1]**, I.6.3.

4.2.8 Definition. Denote by *cld* the family of closed subsets of X/J. Note that if X/J is paracompact, then *cld* is paracompactifying.

4.2.9 Corollary. Let X/J be paracompact. Then the singular J-cohomology of X with coefficients in κ is isomorphic to the cohomology module of the cochain complex $\Gamma_{cld}\mathcal{E}^*$.

Proof. From 4.2.6, 4.2.7, and 4.2.8, we have $H_J^*(X, \kappa) \cong H^*(\Gamma_{cld}\mathcal{E}^*)$.

4.2.10 Recall that κX is a presheaf on X/J defined by $\kappa X(U) = J\text{-CGV}(\check{U}, \kappa)$, taken with the discrete topology, where $U \subseteq X/J$ is an open set and $\mathcal{K}X$ is the induced sheaf. **Our next goal is to show that** $\mathcal{E}^*$, with an appropriate assumption, **is a** *cld*-**acyclic resolution of** $\mathcal{K}X$.

4.2.11 Denote by δ_A^*, δ_B^*, and δ_C^* the coboundary operator of A^*, B^*, and C^* respectively. Now we shall identify the augmentations of the differential presheaves A^*, B^*, and C^*. Note that $\ker(\delta_A^0)$, $\ker(\delta_B^0)$, and $\ker(\delta_C^0)$ are presheaves as kernels of presheaves.

$$\begin{tikzcd}
C^1(\mathrm{U}) \arrow[r] & A^1(\mathrm{U}) \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & B^1(\mathrm{U}) \\
C^0(\mathrm{U}) \arrow[r] \arrow[u, "\delta^0_C"] & A^0(\mathrm{U}) \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] \arrow[u, "\delta^0_A"] & B^0(\mathrm{U}) \arrow[u, "\delta^0_B"'] \\
\ker(\delta^0_C)(\mathrm{U}) \arrow[r] \arrow[u] & \ker(\delta^0_A)(\mathrm{U}) \arrow[r, shift left, "\phi|"] \arrow[r, shift right, "\psi|"'] \arrow[u] & \ker(\delta^0_B)(\mathrm{U}) \arrow[u]
\end{tikzcd}$$

4.2.12. Proposition. Assume that all k-spaces X_a, $J(a,b)$, $a,b \in \mathrm{ob}\, J$, are locally path connected. Let U be an open set in X/J. Then

$$\ker(\delta^0_A)(\mathrm{U}) = \prod_a \mathrm{Top}(t_a^{-1}(\mathrm{U});\ K_a),$$

$$\ker(\delta^0_B)(\mathrm{U}) = \prod_{b,c} \mathrm{Top}(t_c^{-1}(\mathrm{U}) \underline{x} J(b,c);\ K_b), \quad \text{and} \quad \ker(\delta^0_C) = KX.$$

Proof. First we identify $\ker(\delta^*) = \ker\left(S^0(Y;\ G) \xrightarrow{\delta^0} S^1(Y;\ G)\right)$, where Y is a topological space, and G is an abelian group taken with the discrete topology. Direct calculation shows that $\ker(\delta^*)$ consists of (set) functions $Y \longrightarrow G$ which are constant on the path components of Y. If Y is locally path connected, then clearly $\ker(\delta^*)$ consists of functions $Y \longrightarrow G$ which are constant on the components of Y; that is, all continuous functions $Y \longrightarrow G$. We write $\ker(\delta^*) = \mathrm{Top}(Y,\ G)$.

Note that $\delta^0_A = \coprod_a \delta^0_a$, where each δ^0_a is the coboundary operator of a presheaf $S^0(t_a^{-1}(\mathrm{U}); \kappa_a)$. Therefore $\ker(\delta^0_A)(\mathrm{U}) = \coprod_a \ker(\delta^0_a) = \coprod_a \mathrm{Top}(t_a^{-1}(\mathrm{U}); \kappa_a)$. Similarly, $\ker(\delta^0_A)(\mathrm{U}) = \coprod_{b,c} \mathrm{Top}(t_c^{-1}(\mathrm{U})\underline{x}J(b,c); \kappa_b)$.

Note also that the induced maps $\phi|$ and $\psi|$ from $\ker(\delta^0_A)$ to $\ker(\delta^0_B)$ are the restrictions of those from A^0 to B^0. Compare Lemma 4.2.3 to see that the equalizer of $\phi|$ and $\psi|$, which is the kernel of δ^0_C, is J-CGV($\check{\mathrm{U}}$, κ).

4.2.13 Definition ([Spanier], p.330). A presheaf A on a topological space Y is said to be **fine** if, given any locally finite open covering $\mathcal{U}$ of Y, there exists an indexed family $\{e_U\}_{U\in\mathcal{U}}$ of endomorphisms of A such that, for any open V,

(a) For $m \in A(V)$, $e_U(m)|(V-\bar{U}) = 0$.

(b) If V meets only finitely many elements of $\{\bar{U}\}_{U\in\mathcal{U}}$, then for $m \in A(V)$, $m = \sum_{U\in\mathcal{U}} e_U(m)$.

4.2.14 Proposition. Let $\mathcal{U}$ be a locally finite open covering of X/J. For each $n \geq 0$ there exist two indexed families of $\{e^1_U\}_{U\in\mathcal{U}}$ and $\{e^2_U\}_{U\in\mathcal{U}}$ of endomorphisms of A^n and B^n respectively, both satisfying the conditions of 4.2.13 and such that for any open set V the following diagram commutes.

$$\begin{tikzcd} A^n(V) \arrow[r, shift left, "\phi^V"] \arrow[r, shift right, "\psi^V"'] & B^n(V) \\ A^n(V) \arrow[u, "e^1_U"] \arrow[r, shift left, "\phi^V"] \arrow[r, shift right, "\psi^V"'] & B^n(V) \arrow[u, "e^2_U"'] \end{tikzcd}$$

Proof. Compare [Spanier], 6.8, Example 2. Let $\mathcal{U}$ be a locally finite open covering of X/J. For every $x \in X/J$ we choose $U_x \in \mathcal{U}$ such that $x \in U_x$.

Let $a,b \in$ ob J. Note that a family $\{\check{U}_a = t_a^{-1}(U)\}_{U \in \mathcal{U}}$ is a locally finite open covering of X_a, and a family $\{\check{U}_{ab} = \tilde{X}_{ab}^{-1}(t_a^{-1}(U))\}_{U \in \mathcal{U}}$ is a locally finite open covering of $X_b \underline{x} J(a,b)$. To $z \in X_a$ assign $\check{U}_{a_z} = t_a^{-1}(U_x)$, where $x = t_a(z)$. Similarly, to $w \in X_a$ assign $\check{U}_{ab_w} = \tilde{X}_{ab}^{-1}(t_a^{-1}(U_x))$, where $x = t_a(\tilde{X}_{ab}(w))$. First we define families $\{E_{\check{U}_a}\}_{U \in \mathcal{U}}$ and $\{E_{\check{U}_{ab}}\}_{U \in \mathcal{U}}$ of endomorphisms for presheaves $S^n(t_a^{-1}(\_), K_a)$ and $S^n(t_b^{-1}(\_)\underline{x}J(a,b), K_a)$, respectively.

Endomorphisms $E_{\check{U}_a}: S^n(t_a^{-1}(V), K_a) \longrightarrow S^n(t_a^{-1}(V), K_a)$ and

$E_{\check{U}_{ab}}: S^n(t_b^{-1}(V)\underline{x}J(a,b), K_a) \longrightarrow S^n(t_b^{-1}(V)\underline{x}J(a,b), K_a)$ are defined by

$$(E_{\check{U}_a}f)(\sigma) = \begin{cases} f(\sigma) & \check{U}_a = \check{U}_{a\sigma(p_0)} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} f(\sigma) & U = U_{t_a(\sigma(p_0))} \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$(E_{\check{U}_{ab}}f)(\sigma) = \begin{cases} f(\sigma) & \check{U}_{ab} = \check{U}_{ab\sigma(p_0)} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} f(\sigma) & U = U_{t_a(\tilde{X}_{ab}(\sigma(p_0)))} \\ 0 & \text{otherwise} \end{cases},$$

where p_0 is the first vertex in the standard n-simplex Δ_n. Then both families satisfy conditions in 4.2.14 (see [Spanier], 6.8 Example 2). For an open set V define $e^1_U : A^n(V) \longrightarrow A^n(V)$ and $e^2_U : B^n(V) \longrightarrow B^n(V)$ by $e^1_U(\{f_a\})_c = E_{\check{U}c}(f_c)$ and $e^2_U(\{f_{bc}\})_{ad} = E_{\check{U}a}{}^d(f_{ad})$.

We shall show that $\{e^1_U\}_{U\in\mathcal{U}}$ satisfies conditions (a) and (b) of 4.2.14. Let $f = \{f_a\} \in A^n(V)$. Then $\left(e^1_U(\{f_a\})|V\text{-}\bar{U}\right)_c = E_{\check{U}c}(f_c)|V\text{-}\bar{U} = 0$. Let V meet finitely many elements of $\{\bar{U}\}_{U\in\mathcal{U}}$ only. Then $\left(\sum_{U\in\mathcal{U}} e^1_U(\{f_a\})\right)_c = \sum_{U\in\mathcal{U}} E_{\check{U}}(f_c) = f_c$ as required. Similarly for $\{e^2_U\}_{U\in\mathcal{U}}$. To prove the last assertion we show that $\phi^U_{ab} \circ E_{\check{U}a} = E_{\check{U}ab} \circ \phi^U_{ab}$ and $\psi^U_{ab} \circ E_{\check{U}} = E_{\check{U}ab} \circ \psi^U_{ab}$. Specifically

$$\phi^U_{ab}\left(E_{\check{U}a}(f_a)\right)(\sigma) = E_{\check{U}a}(f_a)\left(\tilde{X}_{ab\#}(\sigma)\right) = \begin{cases} f_a\left(\tilde{X}_{ab\#}(\sigma)\right) & U = U_{t_a(\tilde{X}_{ab}(\sigma(p_0)))} \\ 0 & \text{otherwise} \end{cases}$$ and

$$E_{\check{U}ab}\left(\phi^U_{ab}(f_a)\right)(\sigma) = \begin{cases} \phi^U_{ab}(f_a)(\sigma) & U = U_{t_a(\tilde{X}_{ab}(\sigma(p_0)))} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} f_a\left(\tilde{X}_{ab\#}(\sigma)\right) & U = U_{t_a(\tilde{X}_{ab}(\sigma(p_0)))} \\ 0 & \text{otherwise} \end{cases}.$$

Also $\psi^U_{ab}\left(E_{\check{U}}(f_b)\right)(\sigma,\eta) = \check{\kappa}(\eta)\left(E_{\check{U}}(f_b)(\sigma)\right)$

$$= \begin{cases} \check{\kappa}(\eta)\left(f_b(\sigma)\right) & U = U_{t_b(\sigma(p_0))} \\ 0 & \text{otherwise} \end{cases}$$ and

$$E_{\check{U}ab}\left(\psi^U_{ab}(f_b)\right)(\sigma,\eta) = \begin{cases} \left(\psi^U_{ab}(f_b)\right)(\sigma,\eta) & U = U_{t_a(\tilde{X}_{ab}(\sigma(p_0),\eta(p_0)))} \\ 0 & \text{otherwise} \end{cases} =$$

$$\begin{cases} \check{\kappa}(\eta)\left(f_b(\sigma)\right) & U = U_{t_a(\tilde{X}_{ab}(\sigma(p_0),\eta(p_0)))} \\ 0 & \text{otherwise} \end{cases}.$$

Since in general $t_a(\tilde{X}_{ab}(x,g)) = t_a(X_{ab}(g)(x)) = t_b(x)$, from the colimit X/J properties, the required equality holds.

4.2.15 Proposition. Each C^n, $n \geq 0$, is a fine presheaf.

Proof. Let $\mathcal{U}$ be a locally finite open covering of X/J. From 2.16 there exists two specific families of endomorphisms, $\{e^1_U\}_{U \in \mathcal{U}}$ for A^n and $\{e^2_U\}_{U \in \mathcal{U}}$ for B^n. Commutativity in the diagram in 4.2.14 assures that each e^1_U, $U \in \mathcal{U}$, restricts to C^n with the required properties.

4.2.16 Corollary. Each $\mathscr{C}^n$, $n \geq 0$, is ϕ-acyclic if ϕ is paracompactifying.

Proof. From [Spanier 1], 6.8.3(c), C^n is a fine presheaf. The claim follows from a corollary in [Swan], p.75.

4.2.17 Definition (compare [Piacenza 1], p. 11). A diagram $X: J^{op} \longrightarrow$ **CGV** is said to be ***J*-cohomologically locally connected**, abbreviated *J–clc* if the complex of sheaves

$$O \longrightarrow \mathcal{K}X \longrightarrow \mathscr{C}^*(X; \kappa)$$

is exact for every coefficient system $\kappa: J^{op} \longrightarrow \mathbf{Mod}_R$.

4.2.18 Proposition. Consider the following conditions on $X \in$ ob *J*–**CGV**:

(1) If x_0 is a point of X/J, then $\{x_0\}^{\vee}$ is isomorphic to D_j for some $j \in$ ob J.

(2) Every point of X/J is a local *J*–NDR.

Then X is *J–clc*.

Proof. (Compare [Piacenza 1], p. 11) The sequence of sheaves $\bar{\mathscr{C}}^*$: $O \longrightarrow \mathcal{K}X \longrightarrow \mathscr{C}^*(X; \kappa)$ on X/J is exact iff for every $x \in X/J$ the corresponding sequence of stalks is exact

$$(*) \qquad 0 \longrightarrow \mathcal{K}X_x \longrightarrow \mathscr{C}^0_x \xrightarrow{\delta^0} \mathscr{C}^1_x \xrightarrow{\delta^1} \mathscr{C}^2_x \longrightarrow \dots .$$

Observe that $\mathcal{K}X_x \cong \varinjlim_{x \in \mathtt{U}} \mathbb{K}X(\mathtt{U})$, $\mathcal{C}^n_x \cong \varinjlim_{x \in \mathtt{U}} C^n_J(\check{\mathtt{U}}; \mathbb{K})$, where $\mathtt{U}$ ranges over the neighborhoods of x. Since homology commutes with direct limits, the sequence (*) is exact iff $\varinjlim_{x \in \mathtt{U}} \tilde{H}^*_J(\check{\mathtt{U}}; \mathbb{K}) = 0$ for all $x \in X/J$, where $\tilde{H}^*_J$ is the cohomology of the augmented cochain $\bar{C}^*(\mathtt{U})$: $0 \longrightarrow \mathbb{K}X(\mathtt{U}) \longrightarrow C^*(\mathtt{U})$.

By applying the homotopy and dimension axioms to (1) and (2) we obtain the following claim. For each $x \in X/J$ and for each neighborhood $\mathtt{U}$ of x there is a neighborhood $\mathtt{V}$ of x, $\mathtt{V} \subseteq \mathtt{U}$, such that $\tilde{H}^*_J(\check{\mathtt{V}}; \mathbb{K}) \longrightarrow \tilde{H}^*_J(\check{\mathtt{U}}; \mathbb{K})$ is trivial (we can choose $\mathtt{V}$ such that $\tilde{H}^*_J(\check{\mathtt{V}}; \mathbb{K}) \cong \tilde{H}^*_J(\{x_0\}^{\vee}; \mathbb{K}) \cong \tilde{H}^*_J(D_j; \mathbb{K}) = 0$ for some j). Therefore the condition $\varinjlim_{x \in \mathtt{U}} \tilde{H}^*_J(\check{\mathtt{U}}; \mathbb{K}) = 0$, for all $x \in X/J$, is satisfied.

Observe that by the Local Contractibility Theorem 1.6.12 we have:

4.2.19 Corollary. *J-CW* complexes are *J-clc*.

4.2.20 The Comparison Theorem. If $X \in$ ob *J-CGV* is *J-clc*, with all X_a, $J(a,b)$, $a,b \in$ ob J, locally path connected, and with X/J paracompact, then there is a natural isomorphism

$$H^*_J(X; \mathbb{K}) \cong {}_{\mathcal{S}h}H^*_J(X; \mathbb{K}).$$

between the singular and sheaf J-cohomology of X with coefficients in $\mathbb{K}$.

Proof. (Compare **[Piacenza 1]**, Theorem 3.11) If X is *J-clc,* then $\mathcal{C}^*$ is an *cld*-acyclic resolution of $\mathcal{K}X$, provided X/J is paracompact. Therefore, the cohomology module of the cochain complex $\Gamma_{cld}\mathcal{C}^*$ gives us the classical sheaf cohomology $H^*(X/J; \mathcal{K}X) = {}_{\mathcal{S}h}H^*_J(X; \mathbb{K})$ (**[Bredon 1]**, Theorem 4.1, p.34), but that cohomology module is, by Corollary 4.2.9, isomorphic to the singular *J*–cohomology of X with coefficients in κ.

4.2.21 Corollary. The comparison theorem holds for any diagram which is a *J–CW* complex or is *J*–homotopy equivalent to a *J–CW* complex.

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